On Timelike Excitations in the Relativistic Harmonic Oscillator

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Relativistic Harmonic Oscillator

Hamiltonian approach — formally similar to nonrelativistic oscillator

$$\left[\frac{p^2}{2m} + V(x)\right] \psi(x) = \kappa\ \psi(x) \qquad (1)$$

$$p^2 = \eta_{\mu\nu} p^\mu p^\nu \qquad V(x) = \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m\omega^2 \left(\mathbf{x}^2 - t^2\right) \qquad \eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)$$

Three related but different approaches

- Feynman, Kislinger, Ravndal, *Current Matrix Elements from a Relativistic Quark Model* (1971)
- Y. S. Kim and M. Noz, *Covariant Harmonic Oscillators and the Quark Model* (1973)
- Arshansky and Horwitz, *The Quantum Relativistic Two-Body Bound State* (1989)

Issues requiring care and attention

- Lorentz covariance of states
- Indefinite spectrum for timelike separation $(x^2 < 0)$
- Normalizability of states

The approach of Feynman et. al.

Apply methods that work well in 3D nonrelativistic case

Dimensionless coordinates

$$q^\mu = \sqrt{m\omega}\ x^\mu \qquad \pi^\mu = \frac{1}{\sqrt{m\omega}} p^\mu \tag{2}$$

Dirac's factorization of 1D Hamiltonian into first order operators

$$K = \omega\left(N + \tfrac{1}{2}\right) = \omega\left(\bar{a}a + \tfrac{1}{2}\right) \qquad a = \tfrac{1}{\sqrt{2}}\left(q + i\pi\right) \qquad \bar{a} = \tfrac{1}{\sqrt{2}}\left(q - i\pi\right)$$

Separation of variables in Cartesian coordinates

$$\left[a^i, \bar{a}^j\right] = \delta^{ij} \longrightarrow \left[a^\mu, \bar{a}^\nu\right] = \eta^{\mu\nu} \qquad |n\rangle \longrightarrow \left|n^0\right\rangle \left|n^1\right\rangle \left|n^2\right\rangle \left|n^3\right\rangle \tag{3}$$

$$K \longrightarrow \omega\eta_{\mu\nu}\left(\bar{a}^\mu a^\nu + \tfrac{1}{2}\eta^{\mu\nu}\right) = \omega\left(-N^0 + N^1 + N^2 + N^3 + 2\right) \tag{4}$$

Scalar ground state function

$$\psi_0\left(\mathbf{q}\right) = e^{-\left(\delta_{ij}q^i q^j\right)/2} \longrightarrow \psi_0\left(q\right) = e^{-\left(\eta_{\mu\nu}q^\mu q^\nu\right)/2} \tag{5}$$

Ground state annihilated by lowering operators

$$a^\mu\ \psi_0\left(q\right) = 0 \tag{6}$$

Closer look at covariant number representation

Operators

$$a^\mu = \frac{1}{\sqrt{2}}\left(q^\mu + i\pi^\mu\right) \qquad \bar{a}^\mu = \frac{1}{\sqrt{2}}\left(q^\mu - i\pi^\mu\right) \tag{7}$$

Commutation relations

$$[a^\mu, \bar{a}^\nu] = \eta^{\mu\nu} \qquad [N^\mu, \bar{a}^\nu] = \eta^{\mu\nu}\bar{a}^\mu \qquad [N^\mu, a^\nu] = -\eta^{\mu\nu}a^\mu \tag{8}$$

Raising and lowering operators

$$\bar{a}^\mu \left|n\right\rangle = \sqrt{n^\mu + \eta^{\mu\mu}}\ e^{i\phi_+^\mu} \left|n + \eta^{\mu\nu}\mathbf{e}_\nu\right\rangle \qquad a^\mu \left|n\right\rangle = \sqrt{n^\mu}\ e^{i\phi_-^\mu} \left|n - \eta^{\mu\nu}\mathbf{e}_\nu\right\rangle \tag{9}$$

$$\mathbf{e}_\nu = \left(\delta_\nu^0, \delta_\nu^1, \delta_\nu^2, \delta_\nu^3\right)$$

Choice of phase determines structure of ground state

- Feynman: $e^{i\phi_+^0} = e^{i\phi_-^0} = i, \quad e^{i\phi_+^k} = e^{i\phi_-^k} = 1$

$$\bar{a}^0 \left|n\right\rangle = \sqrt{1 - n^0}\ \left|n - \mathbf{e}_0\right\rangle \quad a^0 \left|n\right\rangle = \sqrt{-n^0}\ \left|n + \mathbf{e}_0\right\rangle \longrightarrow n^0 \le 0 \tag{10}$$

- Kim: $e^{i\phi_+^\mu} = e^{i\phi_-^\mu} = 1$

$$\bar{a}^0 \left|n\right\rangle = \sqrt{n^0 - 1}\ \left|n - \mathbf{e}_0\right\rangle \quad a^0 \left|n\right\rangle = \sqrt{n^0}\ \left|n + \mathbf{e}_0\right\rangle \longrightarrow n^0 \ge 1 \tag{11}$$

Norm and spectrum

Feynman phase

- Indefinite norm states

$$\bar{a}^0|0\ 0\ 0\ 0\rangle = \sqrt{1}\ |-1\ 0\ 0\ 0\rangle \qquad a^0|0\ 0\ 0\ 0\rangle = 0 \tag{12}$$

$$\langle n^\mu|\ n^\nu\rangle = \langle 0\ 0\ 0\ 0|a^\mu \bar{a}^\nu|0\ 0\ 0\ 0\rangle = \langle 0\ 0\ 0\ 0|\bar{a}^\nu a^\mu + \eta^{\mu\nu}|0\ 0\ 0\ 0\rangle = \eta^{\mu\nu} \tag{13}$$

- Positive spectrum

$$\langle -n^0\ n^1\ n^2\ n^3|K|-n^0\ n^1\ n^2\ n^3\rangle = \omega(n^0+n^1+n^2+n^3+2) \tag{14}$$

Kim phase

- Positive norm states

$$\bar{a}^0|1\ 0\ 0\ 0\rangle = 0 \qquad a^0|1\ 0\ 0\ 0\rangle = |2\ 0\ 0\ 0\rangle \tag{15}$$

$$\langle n^k|\ n^l\rangle = \langle 1\ 0\ 0|a^k \bar{a}^l|1\ 0\ 0\rangle = \eta^{kl} \tag{16}$$

$$\langle n^0|\ n^0\rangle = \langle 1\ 0\ 0\ 0|\bar{a}^0 a^0|1\ 0\ 0\ 0\rangle = \langle 1\ 0\ 0\ 0|a^0 \bar{a}^0 - \eta^{00}|1\ 0\ 0\ 0\rangle = 1 \tag{17}$$

- Indefinite spectrum

$$\langle n^0\ n^1\ n^2\ n^3|K|n^0\ n^1\ n^2\ n^3\rangle = \omega(-n^0+n^1+n^2+n^3+2) \tag{18}$$

Problems of interpretation in the Feynman approach

Ground state not normalizable

$$\left|\psi_0(q)\right|^2 = e^{-\left(\eta_{\mu\nu}q^\mu q^\nu\right)} = e^{-\left(\mathbf{q}^2 - t^2\right)} \tag{19}$$

Timelike excitations can lead to indefinite norm

$$|\psi|^2 = \langle -n^0\ n^1\ n^2\ n^3\ | -n^0\ n^1\ n^2\ n^3 \rangle = \begin{cases} -1, & n^0 > 0 \\ 1, & n^0 = 0 \end{cases} \tag{20}$$

Ad hoc suppression of timelike excited states

- Only calculate with states satisfying

$$(p \cdot a)\,|\psi\rangle = 0 \tag{21}$$

- Still too many degrees of freedom associated with ground state

$$K\ \psi_0(q) = \eta_{\mu\nu}\eta^{\mu\nu}\frac{1}{2}\omega\ \psi_0(q) = 4 \times \frac{1}{2}\omega\ \psi_0(q) \tag{22}$$

- *Ad hoc* suppression of timelike states $\longrightarrow$ missing states in matrix elements

The approach of Kim and Noz

Normalizable solution for ground state

$$\psi_0(q) = e^{-\left(\mathbf{q}^2+t^2\right)/2} = e^{-\mathbf{q}^2/2} e^{-t^2/2} \tag{23}$$

Equivalent to choice of phase: $e^{i\phi_+^\mu} = e^{i\phi_-^\mu} = 1$

$$\bar{a}^0 |n\rangle = \sqrt{n^0 - 1}\ |n - \mathbf{e}_0\rangle \quad a^0 |n\rangle = \sqrt{n^0}\ |n + \mathbf{e}_0\rangle \longrightarrow n^0 \geq 1 \tag{24}$$

- Trial solution $\phi_0(q) = H(q) e^{-q^2/2}$

$$a^k \varphi_0(q) = \frac{1}{\sqrt{2}} \left(x^k + \partial_k\right) H(q) e^{-\left(\mathbf{q}^2-t^2\right)/2} = \frac{1}{\sqrt{2}} \partial_k H(q) = 0 \tag{25}$$

$$\bar{a}^0 \varphi_0(q) = \frac{1}{\sqrt{2}} (t + \partial_t) H(q) e^{-\left(\mathbf{q}^2-t^2\right)/2} = \frac{1}{\sqrt{2}} (2t + \partial_t) H(q) = 0 \tag{26}$$

- Kim-Noz ground state is solution to first order annihilation equations

$$H(q) = e^{-t^2} \longrightarrow \varphi_0(q) = e^{-\left(\mathbf{q}^2+t^2\right)/2} \tag{27}$$

Covariant Bound States

Many-body formalism — Horwitz and Piron (1973)

$$i\frac{\partial}{\partial\tau}\psi\left(x_1, x_2, \tau\right) = \left[\frac{\left(p_1\right)^2}{2M_1} + \frac{\left(p_2\right)^2}{2M_2} + V\left(x_1, x_2\right)\right]\psi\left(x_1, x_2, \tau\right) \quad (28)$$

Central force problem — Horwitz and Arshansky (1989)

$$\frac{\left(p_1\right)^2}{2M_1} + \frac{\left(p_2\right)^2}{2M_2} + V\left(x_1, x_2\right) = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V\left(x^2\right) \quad (29)$$

$$P^\mu = p_1^\mu + p_2^\mu \qquad M = M_1 + M_2$$

$$p^\mu = \left(M_2 p_1^\mu - M_1 p_2^\mu\right)/M \qquad m = M_1 M_2 / M$$

$$x = x_1 - x_2 \qquad x^2 = \left(\mathbf{x}_1 - \mathbf{x}_2\right)^2 - \left(t_1 - t_2\right)^2 \qquad x^2 \to \mathbf{x}^2 = r^2 \text{ in Galilean limit}$$

Effective one-body oscillator eigenvalue problem

$$\psi\left(x, \tau\right) = \psi\left(x\right) e^{-i\kappa\tau} \qquad \left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)\psi\left(x\right) = \kappa\ \psi\left(x\right) \quad (30)$$

A priori spacelike support

Require spacelike separation

$$K\ \psi\left(q\right)=\left(\frac{p^{2}}{2m}+\frac{1}{2}m\omega^{2}x^{2}\right)\psi\left(x\right)=\kappa\ \psi\left(x\right)\qquad x^{2}=\mathbf{x}^{2}-t^{2}>0 \tag{31}$$

Virial theorem $\Rightarrow \langle K\rangle > 0$

$$\left\langle\frac{p^{2}}{2M}\right\rangle=\frac{1}{2}\left\langle x^{\mu}\partial_{\mu}V\left(x\right)\right\rangle=\frac{1}{2}\left\langle m\omega^{2}x^{2}\right\rangle=\left\langle V\left(x\right)\right\rangle>0 \tag{32}$$

No obvious way to realize nonholonomic constraint in Cartesian coordinates
Hyperspherical parameterization

$$x=\rho\hat{x}\qquad\rho=\sqrt{\mathbf{x}^{2}-t^{2}}\qquad\hat{x}^{2}=\hat{\mathbf{x}}^{2}-\hat{t}^{2}=1 \tag{33}$$

O(3,1) bound state solutions with $\langle K\rangle > 0$ and $K\ \psi_0 = \frac{3}{2}\omega\ \psi_0$

How do these states appear in the number representation?
How do creation/annihilation operators act on these states?
How are the timelike occupation number modes suppressed?

Program

Study O(3) nonrelativistic and O(2,1) covariant cases

- Symmetries of same dimension permits easier comparison
- O(3,1) case involves induced representation of O(3,1) over O(2,1)

Solve eigenvalue equation $K\ \psi = \kappa\ \psi$ in (hyper)spherical coordinates

- Maximal commuting operator set — K, Casimir operator, one generator
- Separation of variables exploiting symmetry operators
- Characterize states by eigenvalues of commuting operators

Extract eigenvalue information from number states

- Express Hamiltonian and symmetry operators in terms of operators a^μ and $\bar{a}^\mu$
- Study how number states participate in (hyper)spherical eigenstates

Results

- O(3) oscillator — obtain energy and angular momentum eigenvalues from number representation
- O(2,1) oscillator — hyperspherical solution not related to number states by any unitary transformation

Harmonic Oscillator in 3 Dimensions

Solution in polar coordinates

Eigenvalue equation

$$K\ \psi = \frac{\omega}{2}\left[\pi^2 + q^2\right]\psi = E\psi \tag{34}$$

Coordinates

$$q = r\left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right) \longrightarrow q^2 = r^2 \tag{35}$$

O(3) generators and Casimir operator

$$L^i = \frac{1}{2}\varepsilon^{ijk}\left(q^k\pi^l - q^l\pi^k\right) = \varepsilon^{ijk}q_j\pi_k \qquad \left[\mathbf{L}^2, K\right] = \left[L^3, K\right] = 0 \tag{36}$$

$$L^3 = -i\partial_\phi \qquad \mathbf{L}^2 = -\partial_\theta^2 - \frac{\cos\theta}{\sin\theta}\partial_\theta + \frac{1}{\sin^2\theta}\left(L^3\right)^2 \tag{37}$$

Momentum

$$\pi^2 = -\nabla^2 = -\partial_r^2 - \frac{2}{r}\partial_r + \frac{1}{r^2}\mathbf{L}^2 \tag{38}$$

Solution

$$\psi_{nlm} = \frac{1}{N_{nl}}e^{-\frac{r^2}{2}}r^l L_n^{l+\frac{1}{2}}\left(r^2\right)Y_{lm}\left(\theta,\phi\right) \longrightarrow E = \omega\left(2n + l + \frac{3}{2}\right) \tag{39}$$

Operators

$$a^k = \frac{1}{\sqrt{2}}\left(q^k + i\pi^k\right) \qquad \bar{a}^k = \frac{1}{\sqrt{2}}\left(q^k - i\pi^k\right) \tag{40}$$

$$\left[a^k, \bar{a}^j\right] = \delta^{kj} \qquad k, j = 1, 2, 3 \tag{41}$$

Hamiltonian

$$\begin{aligned} K &= \omega\left(\bar{a}^k a_k + \frac{1}{2}\delta^k_k\right) = \omega\left(\bar{a}\cdot a + \frac{3}{2}\right) = \omega\left(N^1 + N^2 + N^3 + \frac{3}{2}\right) \\ &= \omega\left(\mathbf{N} + \frac{3}{2}\right) \end{aligned} \tag{42}$$

States

$$N^k\left|n\right\rangle = \bar{a}^k a^k\left|n\right\rangle = n^k\left|n\right\rangle = n^k\left|n^1\ n^2\ n^3\right\rangle \qquad K\ \left|n\right\rangle = \omega\left(n + 3/2\right)\left|n\right\rangle \tag{43}$$

$$\bar{a}^k\left|n\right\rangle = \sqrt{n^k + \delta^{kk}}\left|n + \delta^{kj}\mathbf{e}_j\right\rangle \qquad a^k\left|n\right\rangle = \sqrt{n^k}\left|n - \delta^{kj}\mathbf{e}_j\right\rangle \tag{44}$$

$$\mathbf{e}_j = \left(\delta^1_j, \delta^2_j, \delta^3_j\right) \tag{45}$$

Spherical operators

$$a_\pm = a^1 \pm i a^2 \qquad \bar{a}_\pm = \bar{a}^1 \pm i \bar{a}^2 \tag{46}$$

$$[a_+, \bar{a}_+] = [a_-, \bar{a}_-] = 0 \qquad [a_+, \bar{a}_-] = [a_-, \bar{a}_+] = 2 \tag{47}$$

Number operator

$$\mathbf{N} = \frac{1}{2} \left(\bar{a}_+ a_- + \bar{a}_- a_+ \right) + \bar{a}^3 a^3 \tag{48}$$

$$K = \mathbf{N} + \frac{3}{2} = \frac{1}{2} \left(\bar{a}_+ a_- + \bar{a}_- a_+ \right) + \bar{a}^3 a^3 + \frac{3}{2} \tag{49}$$

Angular momenta

$$\mathbf{L}^2 = \mathbf{N}^2 + \mathbf{N} - (\bar{a} \cdot \bar{a}) (a \cdot a) \tag{50}$$

$$L^3 = \frac{1}{2} \left(\bar{a}_+ a_- - \bar{a}_- a_+ \right) \tag{51}$$

Commutation relations

$$\left[N^3, a_\pm\right] = \left[N^3, \bar{a}_\pm\right] = \left[N^3, L^3\right] = \left[a^3, L^3\right] = \left[\bar{a}^3, L^3\right] = 0 \tag{52}$$

$$\left[\mathbf{L}^2, N^i\right] \neq 0 \qquad \left[L_3, N^1\right] \neq 0 \qquad \left[L_3, N^2\right] \neq 0 \tag{53}$$

L_3 is block diagonal in n and n^3

$$\left\langle n^1 \; n^2 \; n^3 \right| L_3 \left| n^{1\prime} \; n^{2\prime} \; n^{3\prime} \right\rangle = c\left(n^1, n^2, n^3\right) \delta^{nn'} \delta^{n^3 n^{3\prime}} \tag{54}$$

L_3 is Hermitian and preserves $n = n^1 + n^2 + n^3$ and n^3

$$\begin{aligned} L_3 \left| n^1 \; n^2 \; n^3 \right\rangle &= i\sqrt{n^1\left(n^2+1\right)} \left| n^1 - 1 \; n^2 + 1 \; n^3 \right\rangle \\ &\quad - i\sqrt{\left(n^1+1\right) n^2} \left| n^1 + 1 \; n^2 - 1 \; n^3 \right\rangle \end{aligned} \tag{55}$$

Harmonic Oscillator in 3 Dimensions

Multiplicity of states

Label states $\left|n^1\ n^2\ n^3\right\rangle$ by n and n^3

$$\left|k \quad \left(n - n^3\right) - k \quad n^3\right\rangle, \quad k = 0, 1, ..., \left(n - n^3\right), \quad n^3 = 0, 1, ..., n \tag{56}$$

Multiplicity of states for given n

$$\sum_{n^3=0}^{n} \left(n - n^3\right) + 1 = \frac{(n+1)(n+2)}{2} \tag{57}$$

Define states

$$\zeta_{\alpha\beta\gamma} = \frac{1}{N_{\alpha\beta\gamma}} \left(\bar{a}_+\right)^{\alpha} \left(\bar{a}_-\right)^{\beta} \left(\bar{a}^3\right)^{\gamma} |0\rangle \tag{58}$$

where $N^{\alpha\beta\gamma}$ is a normalization factor, and

$$\alpha + \beta + \gamma = n \tag{59}$$

On these states

$$L^3 \zeta_{\alpha\beta\gamma} = \frac{1}{2} \frac{1}{N_{\alpha\beta\gamma}} \left(\bar{a}_+ a_- - \bar{a}_- a_+\right) \left(\bar{a}_+\right)^{\alpha} \left(\bar{a}_-\right)^{\beta} \left(\bar{a}_3\right)^{\gamma} |0\rangle = (\alpha - \beta)\, \zeta_{\alpha\beta\gamma} \tag{60}$$

The total angular momentum

$$\mathbf{L}^2\zeta_{\alpha\beta\gamma} = \frac{1}{N_{\alpha\beta\gamma}}\mathbf{N}\left(\mathbf{N}+1\right)\zeta_{\alpha\beta\gamma} - \frac{1}{N_{\alpha\beta\gamma}}\left(\bar{a}\cdot\bar{a}\right)\left(a\cdot a\right)\zeta_{\alpha\beta\gamma} \tag{61}$$

On the states $\zeta_{\alpha\beta\gamma}$

$$\begin{aligned} \mathbf{L}^2\zeta_{\alpha\beta\gamma} &= \left[\left(\alpha+\beta+\gamma\right)\left(\alpha+\beta+\gamma+1\right) - 4\alpha\beta - \gamma\left(\gamma-1\right)\right]\zeta_{\alpha\beta\gamma} \\ &\quad - \frac{N_{(\alpha+1)(\beta+1)(\gamma-2)}}{N_{\alpha\beta\gamma}}\gamma\left(\gamma-1\right)\zeta_{(\alpha+1)(\beta+1)(\gamma-2)} \\ &\quad - \frac{N_{(\alpha-1)(\beta-1)(\gamma+2)}}{N_{\alpha\beta\gamma}}4\alpha\beta\zeta_{(\alpha-1)(\beta-1)(\gamma+2)} \end{aligned} \tag{62}$$

Harmonic Oscillator in 3 Dimensions

Maximum angular momentum for number state

States $\zeta_{\alpha\beta\gamma}$ have $n = \alpha + \beta + \gamma$ and $m = \alpha - \beta$ but are mixtures of l-states

The cases $(\alpha, \beta, \gamma) = (n, 0, 0)$ and $(\alpha, \beta, \gamma) = (0, n, 0)$ have eigenvalues

$$\mathbf{L}^2 \zeta_{n00} = n\,(n+1)\,\zeta_{n00} \qquad L^3 \zeta_{n00} = n \zeta_{n00} \tag{63}$$

$$\mathbf{L}^2 \zeta_{0n0} = n\,(n+1)\,\zeta_{0n0} \qquad L^3 \zeta_{0n0} = -n \zeta_{n00} \tag{64}$$

The allowed eigenvalues of L^3

$$m = \alpha - \beta = -l, -l+1, ..., l-1, l \tag{65}$$

are consistent with

$$\alpha, \beta = 0, 1, ..., l_{\max} = n \tag{66}$$

$\mathbf{L}^2$ mixes (α, β, γ)-states with $(\alpha \pm 1, \beta \pm 1, \gamma \mp 2)$-states

The mixed states have the same eigenvalues of L^3

$$m = \alpha - \beta = (\alpha \pm 1) - (\beta \pm 1) \tag{67}$$

The angular momentum content of the states $\zeta_{\alpha\beta\gamma}$ is

$$l = n, n-2, n-4, ..., n - 2 \cdot \mathrm{int}\left(\frac{n}{2}\right) = \begin{cases} l_{\max}, l_{\max} - 2, ..., 0 & l_{\max} \text{ even} \\ l_{\max}, l_{\max} - 2, ..., 1 & l_{\max} \text{ odd} \end{cases} \tag{68}$$

Since the multiplicity of l-states is $2l + 1$ the total multiplicity of n-states is

$$\sum_{k=0}^{\mathrm{int}(n/2)} [2(2k) + 1] = \frac{(n+1)(n+2)}{2} \tag{69}$$

as required

For given values of $n = \alpha + \beta + \gamma$ and $m = \alpha - \beta$, the possible (α, β, γ)-states are

$$\alpha = m + k, \qquad \beta = k, \qquad \gamma = n - m - 2k, \qquad k = 0, 1, 2, ..., \mathrm{int}\left(\frac{n-m}{2}\right) \tag{70}$$

The on-diagonal elements of $\mathbf{L}^2$ for given n and m are

$$\mathbf{L}^2{}_{\mathrm{on-diagonal}} = (2n - m)(m + 1) + 2k(2n - 4m - 1) - 8k^2 \tag{71}$$

Since $\mathrm{tr}\left(\mathbf{L}^2\right)$ is invariant under diagonalization, the sum of eigenvalues is

$$\begin{aligned} \mathrm{tr}\left(\mathbf{L}^2\right) &= \sum_{k=0}^{\mathrm{int}\left(\frac{n-m}{2}\right)} \mathbf{L}^2{}_{\mathrm{on-diagonal}} \\ &= \begin{cases} \frac{1}{6}n^3 + \frac{3}{4}n^2 + \frac{5}{6}n - \frac{1}{6}m^3 + \frac{1}{4}m^2 + \frac{1}{6}m & , \quad n - m \ \text{even} \\ \frac{1}{6}n^3 + \frac{3}{4}n^2 + \frac{5}{6}n - \frac{1}{6}m^3 - \frac{1}{4}m^2 + \frac{1}{6}m + \frac{1}{4} & , \quad n - m \ \text{odd} \end{cases} \end{aligned} \tag{72}$$

Harmonic Oscillator in 3 Dimensions

Classification of states

For given values of $n = \alpha + \beta + \gamma$ and $m = \alpha - \beta$, the possible l-states are

$$l = n - 2k \qquad k = 0, 1, ..., \text{int}\left(\frac{n-m}{2}\right) \tag{73}$$

Therefore, for given n and m the sum of eigenvalues is

$$\begin{aligned} \text{tr}\left(\mathbf{L}^2\right) &= \sum_{k=0}^{\text{int}\left(\frac{n-m}{2}\right)} (n-2k)(n-2k+1) \\ &= \begin{cases} \frac{1}{6}n^3 + \frac{3}{4}n^2 + \frac{5}{6}n - \frac{1}{6}m^3 + \frac{1}{4}m^2 + \frac{1}{6}m & , \quad n-m \text{ even} \\ \frac{1}{6}n^3 + \frac{3}{4}n^2 + \frac{5}{6}n - \frac{1}{6}m^3 - \frac{1}{4}m^2 + \frac{1}{6}m + \frac{1}{4} & , \quad n-m \text{ odd} \end{cases} \end{aligned} \tag{74}$$

as required

Regarding k as a principal quantum number n_a the total mode number becomes

$$n = 2\ n_a + l \longrightarrow E = \omega\left(2n_a + l + \frac{3}{2}\right) \tag{75}$$

Harmonic Oscillator in 3 Dimensions

Unitary transformation for total mode number $n = 2$

$$\psi_{nlm} = \frac{1}{N_{nl}} e^{-\frac{r^2}{2}} r^l L_n^{l+\frac{1}{2}} \left(r^2\right) Y_{lm}\left(\theta,\phi\right) = \sum C^{nlm}_{n^1 n^2 n^3}\ H_{n^1}\left(r\sin\theta\cos\phi\right) H_{n^2}\left(r\sin\theta\sin\phi\right) H_{n^3}\left(r\cos\theta\right)$$

l	m	Unitary combinations of number states $\left\|n^1\ n^2\ n^3\right\rangle$
0	0	$\frac{1}{\sqrt{3}}\left[\|0\ 0\ 2\rangle + \|0\ 2\ 0\rangle + \|2\ 0\ 0\rangle\right]$
2	-2	$-\frac{1}{2}\|0\ 2\ 0\rangle + \frac{1}{2}\|2\ 0\ 0\rangle + \frac{1}{\sqrt{2}} i\|1\ 1\ 0\rangle$
	-1	$\frac{1}{\sqrt{2}} i\|0\ 1\ 1\rangle + \frac{1}{\sqrt{2}}\|1\ 0\ 1\rangle$
	0	$-\sqrt{\frac{2}{3}}\|0\ 0\ 2\rangle + \frac{1}{\sqrt{6}}\|0\ 2\ 0\rangle + \frac{1}{\sqrt{6}}\|2\ 0\ 0\rangle$
	1	$-\frac{1}{\sqrt{2}} i\|0\ 1\ 1\rangle + \frac{1}{\sqrt{2}}\|1\ 0\ 1\rangle$
	-2	$-\frac{1}{2}\|0\ 2\ 0\rangle + \frac{1}{2}\|2\ 0\ 0\rangle - \frac{1}{\sqrt{2}} i\|1\ 1\ 0\rangle$

Eigenvalue equation

$$K\,\psi = \frac{\omega}{2}\left[\pi^2 + q^2\right]\psi = \kappa\,\psi \tag{76}$$

Coordinates

$$q = (t, x, y) = \rho\,(\sinh\beta, \cosh\beta\cos\phi, \cosh\beta\sin\phi) \longrightarrow q^2 = \rho^2 \tag{77}$$

O(2,1) generators and Casimir operator

$$L = q^1\pi^2 - q^2\pi^1 \qquad A^1 = q^0\pi^1 - q^1\pi^0 \qquad A^2 = q^0\pi^2 - q^2\pi^0 \tag{78}$$

$$L = -i\partial_\phi \qquad \Lambda = L^2 - \mathbf{A}^2 = \partial_\beta^2 + \frac{\sinh\beta}{\cosh\beta}\partial_\beta + \frac{L^2}{\cosh^2\beta} \tag{79}$$

$$\left[L, A^1\right] = iA^2 \qquad \left[L, A^2\right] = -iA^1 \qquad \left[A^1, A^2\right] = -iL \tag{80}$$

Momentum

$$\pi^2 = -\nabla^2 = -\partial_\rho^2 - \frac{2}{\rho}\partial_\rho + \frac{\Lambda}{\rho^2} \tag{81}$$

Two separation parameters

$$\frac{\rho^2}{R(\rho)}\left(-\partial_\rho^2 - \frac{2}{\rho}\partial_\rho + \rho^2 - \varepsilon\right) R(\rho) = -\frac{1}{b(\beta)\,\Phi(\phi)}\Lambda b(\beta)\,\Phi(\phi) = -\Lambda_1 \tag{82}$$

$$\frac{\cosh^2\beta}{b(\beta)}\left(\partial_\beta^2 + \frac{\sinh\beta}{\cosh\beta}\partial_\beta - \Lambda_1\right) b(\beta) = -\frac{1}{\Phi(\phi)} L^2\Phi(\phi) = -\Lambda_2^2 \tag{83}$$

ϕ-equation

$$L^2\Phi(\phi) = -\partial_\phi^2\Phi(\phi) = \Lambda_2^2\Phi(\phi) \quad \longrightarrow \quad \Phi(\phi) = e^{i\Lambda_2\phi} \tag{84}$$

β-equation

$$\left(\partial_\beta^2 + \frac{\sinh\beta}{\cosh\beta}\partial_\beta - \Lambda_1 + \frac{\Lambda_2^2}{\cosh^2\beta}\right) b(\beta) = 0 \tag{85}$$

Harmonic Oscillator in 2+1 Dimensions

Hyperspherical function

Change of variables

$$z = \tanh \beta \qquad b(z) = \left(1 - z^2\right)^{\frac{1}{4}} P(z) \tag{86}$$

Set constants of integration

$$\Lambda_1 = \mu^2 - \frac{1}{4} \qquad \Lambda_2 = \nu + \frac{1}{2} = \mu + k + \frac{1}{2}, \quad k = 0, 1, 2, \ldots \tag{87}$$

β-equation becomes associated Legendre

$$\left[\left(1 - z^2\right) \partial_z^2 - 2z\partial_z + \nu(\nu + 1) - \frac{\mu^2}{1 - z^2}\right] P_\nu^\mu(z) = 0 \tag{88}$$

Eigenfunction for Λ and L

$$\chi_{\mu+k}^{-\mu}(\beta, \phi) = C_{\mu k} \left(1 - z^2\right)^{\frac{1}{4}} P_{\mu+k}^{-\mu}(z)\, e^{i\left(\mu+k+\frac{1}{2}\right)\phi} \tag{89}$$

$$\Lambda\, \chi_{\mu+k}^{-\mu}(\beta, \phi) = \Lambda_1\, \chi_{\mu+k}^{-\mu}(\beta, \phi) \qquad L\, \chi_{\mu+k}^{-\mu}(\beta, \phi) = \Lambda_2\, \chi_{\mu+k}^{-\mu}(\beta, \phi) \tag{90}$$

Boost operators

$$A^{\pm} = A^1 \pm iA^2 = e^{\pm i\phi}\left[-i\left(1-z^2\right)\partial_z \pm z\partial_\phi\right] \tag{91}$$

Action of lowering operator on lowest state

$$A^{-}\chi_{\mu}^{-\mu}\left(\beta,\phi\right) = e^{-i\phi}\left[-i\left(1-z^2\right)\partial_z - z\partial_\phi\right]\left(1-z^2\right)^{\frac{1}{4}} P_{\mu}^{-\mu}\left(z\right)\Phi_{\mu}\left(\phi\right) = 0 \tag{92}$$

Action of raising operator

$$A^{+}\chi_{\mu}^{-\mu}\left(\beta,\phi\right) = C_{\mu k} i\left(2\mu+1\right)\chi_{\mu+1}^{-\mu}\left(\beta,\phi\right) \tag{93}$$

Generally with $C_{\mu k} = \sqrt{\mu\frac{(2\mu+k)!}{k!}}$

$$A^{+}\chi_{\mu+k}^{-\mu}\left(\beta,\phi\right) = i\sqrt{\left(k+1\right)\left(2\mu+k+1\right)}\chi_{\mu+k+1}^{-\mu}\left(\beta,\phi\right) \tag{94}$$

Harmonic Oscillator in 2+1 Dimensions

Solution of radial equation

Radial equation

$$\left(\partial_\rho^2 + \frac{2}{\rho}\partial_\rho - \rho^2 + \varepsilon - \frac{\mu^2 - \frac{1}{4}}{\rho^2}\right) R(\rho) = 0 \qquad (95)$$

Change of variables

$$x = \rho^2 \qquad R(\rho) = e^{-\frac{x}{2}} x^{\frac{2n-1}{4}} L(x) \qquad (96)$$

ρ -equation becomes Laguerre

$$\left[x\frac{\partial^2}{\partial x^2} + [\mu - x + 1]\frac{\partial}{\partial x} + n\right] L_n^\mu(x) = 0 \qquad (97)$$

$$n = \frac{1}{2}\left[\frac{1}{2}\varepsilon - \mu - 1\right] \quad \longrightarrow \quad \kappa = \omega(2n + \mu + 1) \qquad (98)$$

Solution

$$\psi(\rho, \beta, \phi) = N_{n\mu k}\, e^{-\frac{\rho^2}{2}} \rho^{\mu - \frac{1}{2}} L_n^\mu\left(\rho^2\right) \frac{1}{\sqrt{\cosh\beta}} P_{\mu+k}^{-\mu}(\tanh\beta)\, e^{i\left(\mu + k + \frac{1}{2}\right)\phi} \qquad (99)$$

Harmonic Oscillator in 2+1 Dimensions

Ground state function

Taking $n = \mu = k = 0 \Rightarrow P_{\mu+k}^{-\mu} \longrightarrow P_0 = 1,\ L_n^{\mu} \longrightarrow L_0 = 1$

$$\psi_0(\rho, \beta, \phi) = N_0 \frac{1}{\sqrt{\rho \cosh \beta}} e^{-\frac{\rho^2}{2}} e^{i\frac{1}{2}\phi} \tag{100}$$

Ground state energy (mass)

$$\kappa = \omega(2n + \mu + 1) \longrightarrow \omega = 2 \times \left(\tfrac{1}{2}\omega\right) \tag{101}$$

In Cartesian coordinates $q = (t, x, y)$

$$\sqrt{\rho \cosh \beta} = (x^2 + y^2)^{\frac{1}{4}} \qquad \rho^2 = x^2 + y^2 - t^2 \qquad \phi = \arctan\left(\frac{y}{x}\right)$$

$$\psi_0(t, x, y) = N_0 \frac{1}{(x^2 + y^2)^{1/4}} e^{-(x^2+y^2-t^2)/2} e^{\frac{i}{2}\arctan\left(\frac{y}{x}\right)} \tag{102}$$

Satisfies Cartesian eigenvalue equation

$$\tfrac{1}{2}\left(-\partial^{\mu}\partial_{\mu} + q^{\mu}q_{\mu}\right)\psi_0(t, x, y) = \psi_0(t, x, y) \tag{103}$$

Spherical operators

$$a_\pm = a^1 \pm i a^2 \qquad \bar{a}_\pm = \bar{a}^1 \pm i \bar{a}^2 \tag{104}$$

$$[a_+, \bar{a}_+] = [a_-, \bar{a}_-] = 0 \qquad [a_+, \bar{a}_-] = [a_-, \bar{a}_+] = 2 \tag{105}$$

Number operator

$$K = \mathbf{N} + \frac{3}{2} = \frac{1}{2}\left(\bar{a}_+ a_- + \bar{a}_- a_+\right) - \bar{a}^0 a^0 + \frac{3}{2} \tag{106}$$

Casimir operator and angular momentum

$$\Lambda = \mathbf{N}^2 + \mathbf{N} - \left(\bar{a} \cdot \bar{a}\right)\left(a \cdot a\right) \tag{107}$$

$$L = \frac{1}{2}\left(\bar{a}_+ a_- - \bar{a}_- a_+\right) \tag{108}$$

Harmonic Oscillator in 2+1 Dimensions

Covariant number operators on hyperspherical ground state

Hyperspherical ground state in Cartesian coordinates

$$\psi_0(t,x,y) = N_0 \frac{1}{(x^2+y^2)^{1/4}} e^{-(x^2+y^2-t^2)/2} e^{\frac{i}{2}\arctan\left(\frac{y}{x}\right)} = \psi_0(x,y)\, e^{\,t^2/2} \qquad (109)$$

Spherical operator a_- annihilates ψ_0

$$a_-\ \psi_0 = \frac{1}{4}\sqrt{2}N_0 \left[-\frac{x+iy}{(x^2+y^2)^{\frac{5}{4}}} + \frac{x+iy}{(x^2+y^2)^{\frac{5}{4}}} \right] e^{-(x^2+y^2-t^2)/2} e^{\frac{i}{2}\arctan\left(\frac{y}{x}\right)} = 0 \qquad (110)$$

Eigenstate of L

$$L\ \psi_0 = \frac{1}{2}\left(\bar{a}_+ a_- - \bar{a}_- a_+\right) = -\frac{1}{2}\bar{a}_- a_+ \psi_0 = \frac{1}{2}\psi_0 \qquad (111)$$

Timelike operator a^0 annihilates ψ_0

$$a^0\ \psi_0 = \frac{1}{\sqrt{2}}\left(t - \partial_t\right)\left[\psi_0(x,y)\, e^{\,t^2/2}\right] = 0 \qquad (112)$$

Harmonic Oscillator in 2+1 Dimensions

Coherent subspaces

Commutation relations

$$\left[N^0, a_\pm\right] = \left[N^0, \overline{a}_\pm\right] = \left[N^0, L\right] = \left[a^0, L\right] = \left[\overline{a}^0, L\right] = 0 \tag{113}$$

$$[\Lambda, N^\mu] \neq 0 \qquad \left[L, N^1\right] \neq 0 \qquad \left[L, N^2\right] \neq 0 \tag{114}$$

States

$$\Lambda \left| n\ \mu\ \nu \right\rangle = \underbrace{\left(\mu^2 - \frac{1}{4}\right)}_{\Lambda_1} \left| n\ \mu\ \nu \right\rangle \qquad L \left| n\ \mu\ \nu \right\rangle = \underbrace{\left(\nu + \frac{1}{2}\right)}_{\Lambda_2} \left| n\ \mu\ \nu \right\rangle \tag{115}$$

L is Hermitian and preserves $n = n^1 + n^2 - n^0$ and n^0

$$\begin{aligned} L \left| n^0\ n^1\ n^2 \right\rangle &= i\sqrt{n^1\left(n^2+1\right)} \left| n^0\ n^1 - 1\ n^2 + 1 \right\rangle \\ &\quad - i\sqrt{\left(n^1+1\right) n^2} \left| n^0\ n^1 + 1\ n^2 - 1 \right\rangle \end{aligned} \tag{116}$$

Harmonic Oscillator in 2+1 Dimensions

Multiplicity of states

Label states $\left|n^0\ n^1\ n^2\right\rangle$ by n and n^0

$$\left|n^0 \quad \left(n+n^0\right)-k \quad k\right\rangle, \quad k=0,1,...,\left(n+n^0\right), \quad n^0=1,2,3,... \tag{117}$$

- Multiplicity of states for given n and n^0 is $n+n^0+1$
- Multiplicity of states for given n is infinite

Define states

$$\zeta_{\alpha\beta\gamma}=\frac{1}{N_{\gamma\alpha\beta}}\left(a^0\right)^\gamma\left(\bar{a}_+\right)^\alpha\left(\bar{a}_-\right)^\beta\varphi_{100} \qquad \alpha+\beta-\gamma=n \tag{118}$$

On these states

$$L\zeta_{\alpha\beta\gamma}=\frac{1}{2}\frac{1}{N_{\alpha\beta\gamma}}\left(\bar{a}_+a_- - \bar{a}_-a_+\right)\left(a^0\right)^\gamma\left(\bar{a}_+\right)^\alpha\left(\bar{a}_-\right)^\beta\varphi_{100}=\left(\alpha-\beta\right)\zeta_{\alpha\beta\gamma} \tag{119}$$

Cannot satisfy the requirement for the hyperspherical solution

$$\alpha-\beta=\Lambda_2=\nu+\frac{1}{2}=\mu+k+\frac{1}{2} \qquad \alpha,\beta,\mu,k \text{ integer} \tag{120}$$

Conclusion

3D nonrelativistic oscillator

- Separate variables in Cartesian or spherical coordinates
- Obtain equivalent solutions with identical eigenvalues
- Unitary transformation connects the equivalent solutions

2+1 relativistic oscillator

- Separate variables in Cartesian coordinates
  - Equivalent to covariant number representation
  - Timelike excitations lead to indefinite spectrum or indefinite norm
  - Suppression of timelike excitations may require *ad hoc* corrections
- Separate variables in hyperspherical coordinates
  - Permits *a priori* suppression of timelike excitations
  - Solutions with positive definite spectrum and norm
  - Not equivalent to covariant number representation
  - No unitary transformation connects Cartesian and hyperspherical approaches

Thank You