Abraham-Lorentz-Dirac Equation in 5D Stuekelberg Electrodynamics

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Covariant Hamiltonian mechanics

Unconstrained phase space with invariant evolution parameter τ

Stueckelberg (1943), Feynman (1948), Nambu (1950), Schwinger (1951), Horwitz and Piron (1973)

$$S = \int d\tau \ \left[\dot{x}^\mu p_\mu - K(x, p, \tau)\right] \qquad \dot{x}^\mu = \frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}^\mu = \frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu}$$

Classical free particle

$$K = \frac{1}{2m} p^\mu p_\mu \qquad \dot{x}^\mu = \frac{p^\mu}{m} \qquad \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \qquad \dot{x}^2 = \frac{p^2}{m^2} = \mathrm{const}$$

Free particle Schrodinger equation

$$i\partial_\tau \psi(x, \tau) = \frac{1}{2m} p^\mu p_\mu \ \psi(x, \tau)$$

5D gauge invariance under transformations

Saad, Horwitz, Arshansky (1989)

$$\psi \to e^{ie_0 \Lambda(x,\tau)} \psi \qquad a_\mu \to a_\mu + \partial_\mu \Lambda(x, \tau) \qquad a_5 \to a_5 + \partial_\tau \Lambda(x, \tau)$$

$$\left[i\partial_\tau + e_0 a_5(x, \tau)\right] \psi(x, \tau) = \frac{1}{2m} \left[p^\mu - e_0 a^\mu(x, \tau)\right] \left[p_\mu - e_0 a_\mu(x, \tau)\right] \psi(x, \tau)$$

Covariant central force problems

Separable two-body Hamiltonian: CM + relative motion

Horwitz and Piron (1973)

$$K = \frac{1}{2M_1} p_{1\mu} p_1^\mu + \frac{1}{2M_2} p_{2\mu} p_2^\mu + V(x_1, x_2) = \frac{1}{2M} P^\mu P_\mu + \frac{1}{2m} p^\mu p_\mu + V(x_1 - x_2)$$

Generalize classical central force problems

$$a^\mu (x, \tau) = 0 \qquad\qquad a_5 (x, \tau) = V\left(\sqrt{x^\mu x_\mu}\right) = V\left(\sqrt{\mathbf{x}^2 - t^2}\right)$$

Solutions for relativistic Schrodinger equation

Arshansky and Horwitz (1989), Land and Horwitz (1995), Land and Horwitz (2001)

- Scattering and bound states with relativistic corrections to spectra
- Covariant generalization of Zeeman splitting
  Perturbation $a^\mu (x, \tau) = -\frac{1}{2} F^{\mu\nu} x_\nu$, $F^{\mu\nu} = const$, $F^{\mu\nu} F_{\mu\nu} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right) > 0$
- Covariant generalization of Stark splitting plus state ionization
  Perturbations $a^\mu (x, \tau) = -\frac{1}{2} F^{\mu\nu} x_\nu$, $F^{\mu\nu} = const$, $F^{\mu\nu} F_{\mu\nu} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right) < 0$
  $$a^5 (x, \tau) = -\epsilon^\mu \left(x_\mu + n_\mu\right), \ \epsilon^\mu, n_\mu = const$$

Classical electrodynamics

Classical action with formal designations $x^5 = \tau,\ \partial_5 = \partial_\tau$

$$S_{event} = \int d\tau \left[\tfrac{1}{2} m \dot{x}^\mu \dot{x}_\mu + e_0 \dot{x}^\alpha a_\alpha \right] \qquad \mu,\nu = 0,1,2,3 \qquad \alpha,\beta = 0,1,2,3,5$$

Classical Lorentz force

$$m\ddot{x}_\mu = e_0 \left(\partial_\mu a_\alpha - \partial_\alpha a_\mu \right) \dot{x}^\alpha = e_0 f_{\mu\alpha}(x,\tau)\, \dot{x}^\alpha = e_0 f_{\mu\nu}(x,\tau)\, \dot{x}^\nu + e_0 f_{\mu 5}(x,\tau)$$

$$\frac{d}{d\tau}\left(-\tfrac{1}{2} m \dot{x}^2\right) = e_0 \left(\partial_5 a_\alpha - \partial_\alpha a_5 \right) \dot{x}^\alpha = e_0 f_{5\alpha}(x,\tau)\, \dot{x}^\alpha = e_0 f_{5\nu}(x,\tau)\, \dot{x}^\nu$$

Electromagnetic action with formal designation $f^{\mu 5} = g^{55} f^{\mu}{}_{5} = -f^{\mu}{}_{5}$

$$S_{field} = \int d^4x d\tau \ \left\{ e_0 \dot{X}^\alpha a_\alpha(x,\tau)\, \delta^4\left[x - X(\tau)\right] - \tfrac{\lambda}{4} f^{\alpha\beta}(x,\tau)\, f_{\alpha\beta}(x,\tau) \right\}$$

pre-Maxwell field equations with formal O(3,2) covariance

$$\partial_\beta f^{\alpha\beta}(x,\tau) = \tfrac{e_0}{\lambda} j^\alpha(x,\tau) = e j^\alpha(x,\tau) \qquad \partial_{[\alpha} f_{\beta\gamma]} = 0$$

Conserved 5-current

$$j^\alpha(x,\tau) = \dot{X}^\alpha(\tau)\, \delta^4\left[x - X(\tau)\right] \qquad \partial_\mu j^\mu + \partial_\tau j^5 = 0$$

Concatenation

Integrate over τ

$$\left.\begin{array}{c}\partial_\beta f^{\alpha\beta}(x,\tau) = ej^\alpha(x,\tau) \\ \partial_{[\alpha} f_{\beta\gamma]} = 0\end{array}\right\} \xrightarrow[\int d\tau]{} \left\{\begin{array}{c}\partial_\nu F^{\mu\nu}(x) = eJ^\mu(x) \\ \partial_{[\mu} F_{\nu\rho]} = 0\end{array}\right.$$

Field and current

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau f^{\mu\nu}(x,\tau) \qquad J^\mu(x) = \int_{-\infty}^{\infty} d\tau\ \dot{X}^\alpha(\tau)\,\delta^4[x - X(\tau)]$$

Boundary conditions

$$j^5(x,\tau) \to 0 \ \text{ and } \ f^{5\mu}(x,\tau) \to 0 \ \text{ at } \ (x,\tau) \to (x,\pm\infty)$$

Interpretation

Arshansky, Horwitz, Lavie (1983)

- $f^{\alpha\beta}(x,\tau)$ mediates interactions between instantaneous events $x^\mu(\tau)$
- Particle worldline traced out by τ-evolution of event $x^\mu(\tau)$
- Concatenation extracts Maxwell as equilibrium limit of event dynamics
- $\dim\left[e_0 \int d\tau\ a^\mu(x,\tau)\right] = \dim\left[eA^\mu(x)\right] \Rightarrow \dim[e_0] = \dim[\lambda] = \text{time}$

Wave equation and Greens function

Wave equation

$$\partial_\alpha \partial^\alpha a^\beta (x,\tau) = \left(\partial_\mu \partial^\mu - \partial_\tau^2\right) a^\beta (x,\tau) = -e j^\beta (x,\tau)$$

Greens function

Land, Horwitz (1991)

$$\left(\partial_\mu \partial^\mu - \partial_\tau^2\right) G(x,\tau) = -\delta^4(x,\tau)$$

$$G(x,\tau) = -\frac{1}{2\pi}\delta(x^2)\delta(\tau) - \frac{1}{2\pi^2}\frac{\partial}{\partial x^2}\frac{\theta(x^2-\tau^2)}{\sqrt{x^2-\tau^2}} = D(x)\,\delta(\tau) - G_{correlation}(x,\tau)$$

Concatenation

$$\int d\tau\; G(x,\tau) = \int d\tau \Big[D(x)\,\delta(\tau) - G_{correlation}(x,\tau) \Big] = D(x)$$

Interpretation

- Support of $D(x)\,\delta(\tau)$ on lightcone and instantaneous in τ
- Support of $G_{correlation}$ is spacelike $x^2 > \tau^2 \geq 0$
- Spacelike interaction does not contribute to Maxwell potentials

Induced potential

pre-Maxwell potential

$$a^\beta(x,\tau) = -e\int d^4x'd\tau' G\left(x-x',\tau-\tau'\right)\ j^\beta(x',\tau')$$

Concatenated potential

$$\begin{aligned} A^\mu(x) &= -e\int d^4x'd\tau'\left\{\int d\tau\ G\left(x-x',\tau-\tau'\right)\right\}\ j^\beta(x',\tau') \\ &= -e\int d^4x' D\left(x-x'\right)\int d\tau'\ j^\mu(x',\tau') \\ &= -e\int d^4x' D\left(x-x'\right) J^\mu(x') \end{aligned}$$

Nonrelativistic Coulomb problem: source event evolves as $X(\tau) = (\tau, 0, 0, 0)$

$$a^0(x,\tau) = -e\int d^4x'd\tau' D\left(x-x'\right)\delta\left(\tau-\tau'\right)\delta^4\left(x'-\tau'\right) = -\frac{e}{2\pi}\,\delta\left[(x-\tau)^2\right]\theta\left(x^0\right)$$

$$= -\frac{e}{2\pi}\frac{\delta\left(x^0-\tau-R\right)}{|2R|} \quad \xrightarrow[\int d\tau]{} \quad A^0(x) = -\frac{e}{4\pi R}$$

Correct Maxwell potential — wrong microscopic dynamics

Statistical synchronization in field-current interaction

Distribute current density along worldline, preserving concatenation

Land (1997)

$$j^{\alpha}(x,\tau) \longrightarrow j^{\alpha}_{\varphi}(x,\tau) = \int_{-\infty}^{\infty} ds\ \varphi(\tau - s)\, j^{\alpha}(x,s) \qquad \varphi(\tau) = \frac{1}{2\lambda} e^{-|\tau|/\lambda}$$

$$J^{\mu}(x) = \int_{-\infty}^{\infty} d\tau\ j^{\mu}_{\varphi}(x,\tau) = \int_{-\infty}^{\infty} ds\ d\tau\ \varphi(\tau - s)\, j^{\mu}(x,s) = \int_{-\infty}^{\infty} ds\ j^{\mu}(x,s)$$

Induces Yukawa-type potential with correct non-relativistic limit for large λ

$$m\ddot{\mathbf{x}} = -e_0 \nabla \left[a^0(x,\tau) + a^5(x,\tau) \right] \to m\ddot{\mathbf{x}} = -2e_0 \nabla a^0_{\varphi}(x,\tau) = e^2 \nabla \left[\frac{e^{-|\mathbf{x}|/\lambda}}{4\pi |\mathbf{x}|} \right]$$

Interpretation

- Relax τ-synchronization between interacting classical events
- Event $x_{\mu}(\tau)$ interacts with current $j^{\alpha}(\tau + \delta\tau)$ for φ-distributed $\delta\tau$
- φ cuts off photon mass spectrum: take $\Delta m_{\gamma} \simeq 10^{-15}\,\mathrm{eV} \to \lambda \simeq$ 4 seconds
- Limits: $\lambda \to 0$ restores $\varphi(\tau) \to \delta(\tau)$
  $\lambda \to \infty$ restores Maxwell

Lagrangian approach to synchronization

O(3,1)-invariant correction to action

Land (2002)

$$S_{em} \longrightarrow \int d^4x d\tau\ e_0 j^\alpha a_\alpha - \frac{\lambda}{4} f^{\alpha\beta}(x,\tau)\, f_{\alpha\beta}(x,\tau) - \frac{\lambda^3}{4} \partial_\tau f^{\alpha\beta}(x,\tau) \partial_\tau f_{\alpha\beta}(x,\tau)$$

$$= \int d^4x d\tau\ e_0 j^\alpha a_\alpha - \frac{\lambda}{4} \int d^4x\, d\tau\, ds\ f^{\alpha\beta}(x,\tau) \Phi(\tau - s) f_{\alpha\beta}(x,s)$$

Field interaction kernel

$$\Phi(\tau) = \delta(\tau) - \lambda^2 \delta''(\tau) = \frac{1}{2\pi} \int d\kappa \left[1 + (\lambda\kappa)^2 \right] e^{-i\kappa\tau}$$

Inverse function

$$\int_{-\infty}^{\infty} ds\ \Phi(\tau - s) \varphi(s - r) = \delta(\tau - r) \rightarrow \varphi(\tau) = \frac{1}{2\pi} \int d\kappa\ \frac{1}{1 + (\lambda\kappa)^2} = \frac{1}{2\lambda} e^{-|\tau|/\lambda}$$

Euler-Lagrange equations

$$\partial_\beta \int ds\, \Phi(\tau - s) f^{\alpha\beta}(x,s) = e j^\alpha(x,\tau) \xrightarrow[\int \varphi]{} \partial_\beta f^{\alpha\beta}(x,\tau) = e \int ds\ \varphi(\tau - s)\, j^\alpha(x,s)$$

Liénard-Wiechert potential

Event $X^\mu(\tau) \to$ current $j^\mu_\varphi(x,\tau) \to$ potential $a^\alpha(x,\tau)$

Dirac (1938), Barut (1964)

$$
\begin{aligned}
a^\beta(x,\tau) &= -e\int d^4x' d\tau' D(x-x')\,\delta(\tau-\tau')\int ds\ \varphi(\tau'-s)\,\dot{X}^\alpha(s)\,\delta^4\left[x'-X(s)\right] \\
&= -\frac{e}{2\pi}\int ds\ \varphi(\tau-s)\,\dot{X}^\alpha(s)\,\delta\left((x-X(s))^2\right)\theta^{ret} \\
&= -\frac{e}{2\pi}\varphi(\tau-s)\,\frac{\dot{X}^\beta(s)}{2\left(x^\mu-X^\mu(s)\right)u_\mu} = -e\frac{1}{2\lambda}\frac{u^\beta}{4\pi R}e^{-|\tau-s|/\lambda}
\end{aligned}
$$

using identity $\displaystyle\int d\tau f(\tau)\,\delta\left[g(\tau)\right] = \frac{f(s)}{|g'(s)|}$ for $s = g^{-1}(0)$

with

- arbitrary timelike velocity $u^\beta = \dot{X}^\beta(s)$
- $z^\mu = x^\mu - X^\mu(s) \Rightarrow \dot{z}^\mu = -u^\mu$
- retarded time s satisfies $z^2 = 0$ and $\theta^{ret} = \theta\left(x^0 - X^0(s)\right) = 1$
- $R = \frac{1}{2}\frac{d}{ds}(x-X(s))^2 = -z\cdot u > 0$

Field derivatives

Spacetime derivatives

$$\begin{aligned}
\partial^{\mu} a^{\beta}(x,\tau) &= -\frac{e}{2\pi}\int ds\ \varphi(\tau - s)\,\dot{X}^{\alpha}(s)\,\theta^{ret}\partial^{\mu}\delta\left((x - X(s))^2\right) \\
&= \frac{e}{2\pi}\int ds\ \varphi(\tau - s)\dot{X}^{\beta}(s)\,\theta^{ret}\delta'\left[(x - X(s))^2\right]\left[-2\left(x^{\mu} - X^{\mu}(s)\right)\right] \\
&= \frac{e}{2\pi}\int ds\ \varphi(\tau - s)\dot{X}^{\beta}(s)\,\frac{x^{\mu} - X^{\mu}(s)}{\dot{X}(s)\cdot(x - X(s))}\theta^{ret}\frac{d}{ds}\delta\left[(x - X(s))^2\right]
\end{aligned}$$

Integration by parts

$$\begin{aligned}
\partial^{\mu} a^{\beta}(x,\tau) &= -\frac{e}{2\pi}\int ds\ \left[\frac{d}{ds}\varphi(\tau - s)\dot{X}^{\beta}(s)\,\frac{x^{\mu} - X^{\mu}(s)}{\dot{X}(s)\cdot(x - X(s))}\right]\theta^{ret}\delta\left[(x - X(s))^2\right] \\
&= -\frac{e}{4\pi}\frac{1}{R}\frac{d}{ds}\left[\varphi(\tau - s)\frac{z^{\mu}u^{\beta}}{R}\right]
\end{aligned}$$

τ-derivative

$$\partial_{\tau} a_{\mu}(x,\tau) = -e\dot{\varphi}(\tau - s)\,\frac{u_{\mu}}{4\pi R} = e\epsilon(\tau - s)\,\varphi(\tau - s)\,\frac{u_{\mu}}{4\pi\lambda R}$$

Field strengths

Spacetime component

$$f^{\mu\nu} = -\frac{e}{4\pi}\frac{1}{R}\frac{d}{ds}\left[\varphi(\tau - s)\frac{z^\mu u^\nu - z^\nu u^\mu}{R}\right]$$

$$\boxed{\begin{array}{l} R = -u \cdot z > 0 \\ \dot{R} = u^2 - z \cdot \dot{u} \end{array}}$$

$$f^{\mu\nu}_{ret} = e\varphi\left(\tau - s\right)\left[\frac{\left(z^\mu u^\nu - z^\nu u^\mu\right)u^2}{4\pi\left(u \cdot z\right)^3} - \epsilon\left(\tau - s\right)\frac{\left(z^\mu u^\nu - z^\nu u^\mu\right)}{4\pi\lambda\left(u \cdot z\right)^2}\right] \sim \frac{1}{\mathbf{z}^2}$$

$$f^{\mu\nu}_{rad} = e\varphi(\tau - s)\frac{\left(z^\mu \dot{u}^\nu - z^\nu \dot{u}^\mu\right)\left(u \cdot z\right) - \left(z^\mu u^\nu - z^\nu u^\mu\right)\left(\dot{u} \cdot z\right)}{4\pi\left(u \cdot z\right)^3} \sim \frac{1}{|\mathbf{z}|}$$

5-component

$$\partial_\tau a_\mu\left(x, \tau\right) = e\epsilon\left(\tau - s\right)\varphi\left(\tau - s\right)\frac{u_\mu}{4\pi\lambda R} \qquad \partial_\mu a_5 = \frac{e}{4\pi}\frac{1}{R}\frac{d}{ds}\left[\varphi(\tau - s)\frac{z_\mu}{R}\right]$$

$$f^{ret}_{\mu 5} = -e\varphi\left(\tau - s\right)\left[\frac{z_\mu u^2 - u_\mu\left(u \cdot z\right)}{4\pi\left(u \cdot z\right)^3} - \epsilon\left(\tau - s\right)\frac{z_\mu - u_\mu\left(u \cdot z\right)}{4\pi\lambda\left(u \cdot z\right)^2}\right] \sim \frac{1}{\mathbf{z}^2}$$

$$f^{rad}_{\mu 5} = -e\varphi\left(\tau - s\right)\frac{z_\mu\left(\dot{u} \cdot z\right)}{4\pi\left(u \cdot z\right)^3} \sim \frac{1}{|\mathbf{z}|}$$

Electromagnetic two form

Clifford bivector representation

$$f = \frac{1}{2} f^{\mu\nu} \left(\mathbf{e}_\mu \wedge \mathbf{e}_\nu \right) = \frac{1}{2} f^{\mu\nu} \left(\mathbf{e}_\mu \otimes \mathbf{e}_\nu - \mathbf{e}_\nu \otimes \mathbf{e}_\mu \right)$$

Spacetime component

$$f_{spacetime} = -\frac{e}{4\pi} \frac{1}{R} \frac{d}{ds} \left[\varphi(\tau - s) \frac{z \wedge u}{R} \right]$$

$$\begin{aligned} f^{rad}_{spacetime} &= e\varphi(\tau - s) \frac{(z \wedge \dot{u})(u \cdot z) - (z \wedge u)(\dot{u} \cdot z)}{4\pi (u \cdot z)^3} \\ &= e\varphi(\tau - s)\ z \wedge \frac{\dot{u}(u \cdot z) - u(\dot{u} \cdot z)}{4\pi (u \cdot z)^3} \end{aligned}$$

Introduce

$$Q = -\dot{u} \cdot z \qquad w = \dot{u}R - uQ = -\left[\dot{u}(u \cdot z) - u(\dot{u} \cdot z) \right]$$

$$f^{rad}_{spacetime} = e\varphi(\tau - s)\ \frac{z \wedge w}{4\pi R^3}$$

Radiation two form

Formal 5-velocity

$$U = \frac{d}{d\tau}\left(X\left(\tau\right),\tau\right) = \left(\dot{X}\left(\tau\right),1\right) = u + \mathbf{e}_5 \qquad \dot{U} = \frac{d}{d\tau}\left(\dot{X}\left(\tau\right),1\right) = \left(\dot{u},0\right) = \dot{u}$$

5-component of radiation field

$$f_5 = -e\varphi\left(\tau - s\right)\ z \wedge \mathbf{e}_5\ \frac{\left(\dot{u}\cdot z\right)}{4\pi\left(u\cdot z\right)^3} = -e\varphi\left(\tau - s\right)\ \frac{z\wedge\mathbf{e}_5\ Q}{4\pi R^3}$$

Formal 5D two-form

$$\begin{aligned} f &= f_{spacetime} + f_5 \\ &= e\varphi(\tau - s)\ \frac{z\wedge w}{4\pi R^3} - e\varphi\left(\tau - s\right)\ \frac{z\wedge\mathbf{e}_5\ Q}{4\pi R^3} \\ &= e\varphi(\tau - s)\ \frac{z\wedge W}{4\pi R^3} \end{aligned}$$

where

$$W = w - Q\mathbf{e}_5 = \dot{u}R - uQ - \mathbf{e}_5 Q = \dot{U}R - UQ$$

Field invariants

Fields

$$f = e\varphi(\tau - s)\ \frac{z \wedge W}{4\pi R^3} = e\varphi(\tau - s)\ \frac{z \wedge w - z \wedge \mathbf{e}_5\ Q}{4\pi R^3}$$

Noting

$$z^2 = z \cdot \mathbf{e}_5 = 0$$

$$z \cdot w = z \cdot (\dot{u}R - uQ) = (z \cdot \dot{u})\, R - (z \cdot u)\, Q = -QR + RQ = 0$$

$$z \cdot W = z \cdot w - z \cdot \mathbf{e}_5 Q = 0$$

and Clifford identities

$$a \cdot (b \wedge c) = (a \cdot b)\, c - (a \cdot c)\, b$$

$$(a \wedge b) \cdot (a \wedge b) = [(a \wedge b) \cdot a] \cdot b = (a \cdot b)^2 - a^2 b^2$$

Null field satisfies

$$z \cdot f = 0 \ \longrightarrow\ z_\mu f^{\mu\alpha} \ = 0$$

$$-f \cdot f = 0 \ \longrightarrow\ f^{\alpha\beta} f_{\alpha\beta} = f^{\mu\nu} f_{\mu\nu} = f^{5\nu} f_{5\nu} = 0$$

$$\epsilon^{\alpha\beta\gamma\delta\varepsilon} f_{\beta\gamma} f_{\delta\varepsilon} = 0 \ \longrightarrow\ \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} f_{\sigma 5} f_{\rho\nu} = 0$$

Translation invariant electromagnetic action

$$S_{field} = -\frac{\lambda}{4} \int d^4x \, d\tau \, d\tau' \; f^{\alpha\beta}(x,\tau)\Phi(\tau - \tau')f_{\alpha\beta}\left(x,\tau'\right)$$

Conserved tensor

$$T^{\alpha\beta} = -\lambda \left(g^{\alpha\beta} f^{\delta\gamma} f^{\Phi}_{\delta\gamma} - f^{\alpha}{}_{\gamma} f^{\beta\gamma}_{\Phi} \right) \qquad \partial_\beta T^{\alpha\beta} = e f^{\alpha\beta} j_\alpha$$

where

$$f_\Phi\left(x,\tau\right) = \int d\tau' \; \Phi(\tau - \tau') \; f\left(x,\tau'\right)$$

For the radiation field

$$f_\Phi\left(x,\tau\right) = e \int d\tau' \Phi(\tau - \tau') \varphi(\tau' - s) \frac{z \wedge W}{4\pi R^3} = e\delta(\tau - s)\frac{z \wedge W}{4\pi R^3}$$

Null field

$$f^{\alpha\beta} f_{\alpha\beta} = 0 \qquad \Rightarrow \qquad T^{\alpha\beta} = \lambda f^{\alpha}{}_{\gamma} \, f^{\beta\gamma}_{\Phi}$$

Mass-energy-momentum tensor

Components

For bivector

$$A = z \wedge W = \frac{1}{2} A^{\alpha\gamma} \mathbf{e}_\alpha \wedge \mathbf{e}_\gamma \qquad A^{\alpha\gamma} = z^\alpha W^\gamma - z^\gamma W^\alpha$$

Product

$$A^{\alpha}{}_{\gamma} A^{\beta\gamma} = z^2 W^\alpha W^\beta + W^2 z^\alpha z^\beta - z \cdot W \left(z^\alpha W^\beta + z^\alpha W^\beta \right) = W^2 z^\alpha z^\beta$$

with $\delta(\tau - s)\, \varphi(\tau - s) = \delta(\tau - s)\, \varphi(0) = \frac{1}{2\lambda} \delta(\tau - s)$

$$T^{\alpha\beta} = \delta(\tau - s) \frac{1}{2} \left(\frac{e}{4\pi R^3} \right)^2 W^2 z^\alpha z^\beta$$

Separation vector: $z = x - X(s) = [x^\mu - X^\mu(s)]\, \mathbf{e}_\mu \qquad z^5 \equiv 0$

$$T^{\alpha 5} = 0 \qquad\qquad T^{\mu\nu} = \delta(\tau - s) \frac{1}{2} \left(\frac{e}{4\pi R^3} \right)^2 W^2 z^\mu z^\nu$$

Total mass-energy-momentum of particle + field (retarded and radiated)

$$\int d^4x\ \partial_\beta T^{\alpha\beta} = \int d^4x\ \partial_\mu T^{\alpha\mu} + \int d^4x\ \partial_5 T^{\alpha 5} = \tfrac{d}{d\tau} \int d^4x\ T^{5\alpha}$$

$$e_0 \int d^4x\ f^{\alpha\beta}(x,\tau)\, j_\alpha(x,\tau) = e_0 \int d^4x\ f^{\alpha\beta}(x,\tau)\, \dot{X}_\alpha \delta^4(x-X) = e_0 f^{\alpha\beta}(X,\tau)\, \dot{X}_\alpha$$

$$\tfrac{d}{d\tau}\left(\int d^4x\ T^{5\mu} + m\dot{X}^\mu\right) = 0 \qquad \tfrac{d}{d\tau}\left(\int d^4x\ T^{55} - \tfrac{1}{2} m\dot{X}^2\right) = 0$$

For radiation field in region with $j_\alpha(x,\tau) = 0$

$$T^{5\alpha} = 0 \Rightarrow \text{no mass radiation}$$

Spacetime Poynting vector

$$\int d^3x\ d\tau\ \partial_\nu T^{\mu\nu} = \int d^3x\ d\tau\ \partial_0 T^{\mu 0} + \int d^3x\ d\tau\ \partial_i T^{\mu i} = \tfrac{d}{dt} \int d^3x\ d\tau\ T^{0\mu} = 0$$

Oriented along $z = x - X(s)$

$$\int d\tau\ T^{0\mu} = \frac{1}{2}\left(\frac{e}{4\pi R^3}\right)^2 W^2 z^0 z^\nu$$

Vector field picture

Vector components

$$e^i = f^{0i} \qquad b^i = \epsilon^{ijk} f_{jk} \qquad \varepsilon^\mu = f^{5\mu} = \left(\varepsilon^0, \boldsymbol{\varepsilon}\right) = |\boldsymbol{\varepsilon}|\,(1, \hat{\mathbf{z}})$$

Vector field equations

$$\nabla \cdot \mathbf{e} - \partial_\tau \varepsilon^0 = ej^0 \qquad \nabla \times \mathbf{e} + \partial_0 \mathbf{h} = 0 \qquad \nabla \varepsilon^0 - \partial_\tau \mathbf{e} + \partial_0 \boldsymbol{\varepsilon} = 0$$

$$\nabla \times \mathbf{h} - \partial_0 \mathbf{e} - \partial_\tau \boldsymbol{\varepsilon} = e\mathbf{j} \qquad \nabla \cdot \mathbf{h} = 0$$

$$\nabla \cdot \boldsymbol{\varepsilon} + \partial_0 \varepsilon^0 = ej^5 \qquad \nabla \times \boldsymbol{\varepsilon} + \partial_\tau \mathbf{h} = 0$$

Poynting vectors

$$T^{00} = \frac{\lambda}{2}\left[\mathbf{e} \cdot \mathbf{e}^\Phi + \mathbf{b} \cdot \mathbf{b}^\Phi - \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\boldsymbol{\Phi}} - \varepsilon^0 \varepsilon^0_\Phi\right] \qquad T^{0i} = \lambda \left[\mathbf{e} \times \mathbf{b}^\Phi - \varepsilon^0 \boldsymbol{\varepsilon}^{\boldsymbol{\Phi}}\right]^i$$

$$T^{55} = \frac{\lambda}{2}\left[-\mathbf{e} \cdot \mathbf{e}^\Phi + \mathbf{b} \cdot \mathbf{b}^\Phi + \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\boldsymbol{\Phi}} - \varepsilon^0 \varepsilon^0_\Phi\right] \qquad T^{5i} = \lambda \left[\varepsilon^0 \mathbf{e}^\Phi + \boldsymbol{\varepsilon} \times \mathbf{b}^\Phi\right]^i$$

Null field invariants

$$f^{\mu\nu} f_{\mu\nu} = f^{5\nu} f_{5\nu} = 0 \quad \longrightarrow \quad -\mathbf{e} \cdot \mathbf{e}^\Phi + \mathbf{b} \cdot \mathbf{b}^\Phi = \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}^{\boldsymbol{\Phi}} - \varepsilon^0 \varepsilon^0_\Phi = 0$$

$$\epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} f_{\sigma 5} f_{\rho\nu} = 0 \quad \longrightarrow \quad \mathbf{e} \times \mathbf{b} = \left(\mathbf{e} \cdot \mathbf{e}^\Phi\right) \hat{\boldsymbol{\varepsilon}} \qquad \varepsilon^0 \mathbf{e} = -\boldsymbol{\varepsilon} \times \mathbf{b}$$

Plane wave expansion

Fourier transform

$$f^{\alpha\beta}(x,\tau) = \frac{1}{(2\pi)^5}\int d^4k d\kappa \; e^{i(k\cdot x-\kappa\tau)} f^{\alpha\beta}(k,\kappa) \xrightarrow[\int d\tau]{} \frac{1}{(2\pi)^4}\int d^4k \; e^{ik\cdot x} f^{\alpha\beta}(k,0)$$

Wave equations

$$\left(\partial_\mu\partial^\mu - \partial_\tau^2\right) f^{\alpha\beta}(x,\tau) = 0 \quad \Rightarrow \quad k^\alpha k_\alpha = k^\mu k_\mu - \kappa^2 = \mathbf{k}^2 - (k^0)^2 - \kappa^2 = 0$$

Solution to 3-vector equations

$$\mathbf{e} = \mathbf{e}_\perp - \frac{\kappa}{k^0}\,\boldsymbol{\varepsilon}_\parallel \qquad \mathbf{h} = \frac{1}{k^0}\mathbf{k}\times\mathbf{e}_\perp \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\parallel + \frac{\kappa}{k^0}\,\mathbf{e}_\perp \qquad \varepsilon^0 = \frac{1}{k^0}\mathbf{k}\cdot\boldsymbol{\varepsilon}_\parallel$$

Poynting vectors

$$T^{00} = \lambda\left(\mathbf{e}_\perp\cdot\mathbf{e}_\perp^\Phi - \boldsymbol{\varepsilon}_\parallel\cdot\boldsymbol{\varepsilon}_\parallel^\Phi\right) \qquad T^{0i} = \frac{k^i}{k^0}T^{00}$$

$$T^{55} = \left(\frac{\kappa}{k^0}\right)^2 T^{00} \qquad T^{5\mu} = \frac{\kappa}{k^0}\frac{k^\mu}{k^0}T^{00}$$

Radiation field fits equilibrium solution

For on-shell radiation field

$$\kappa = 0 \;\Rightarrow\; \mathbf{k}^2 - (k^0)^2 = 0 \;\Rightarrow\; \mathbf{k}/k^0 = \hat{\mathbf{k}}$$

Solution to 3-vector equations

$$\mathbf{e} = \mathbf{e}_\perp - \frac{\kappa}{k^0}\,\boldsymbol{\varepsilon}_\parallel \xrightarrow[\kappa\to 0]{} \mathbf{e}_\perp \qquad \mathbf{h} = \frac{1}{k^0}\mathbf{k}\times\mathbf{e}_\perp \xrightarrow[\kappa\to 0]{} \hat{\mathbf{k}}\times\mathbf{e}_\perp$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\parallel + \frac{\kappa}{k^0}\,\mathbf{e}_\perp \xrightarrow[\kappa\to 0]{} \boldsymbol{\varepsilon}_\parallel \qquad \varepsilon^0 = \frac{1}{k^0}\mathbf{k}\cdot\boldsymbol{\varepsilon}_\parallel \xrightarrow[\kappa\to 0]{} \hat{\mathbf{k}}\cdot\boldsymbol{\varepsilon}_\parallel$$

Poynting vectors

$$T^{00} = \lambda\left(\mathbf{e}_\perp\cdot\mathbf{e}_\perp^\Phi - \boldsymbol{\varepsilon}_\parallel\cdot\boldsymbol{\varepsilon}_\parallel^\Phi\right) \qquad T^{0i} = \frac{k^i}{k^0}T^{00} \xrightarrow[\kappa\to 0]{} \hat{k}^i T^{00}$$

$$T^{55} = \left(\frac{\kappa}{k^0}\right)^2 T^{00} \xrightarrow[\kappa\to 0]{} 0 \qquad T^{5\mu} = \frac{\kappa}{k^0}\frac{k^\mu}{k^0}T^{00} \xrightarrow[\kappa\to 0]{} 0$$

Angular distribution

From

$$\int d\tau \; T^{0\mu} = \frac{1}{2}\left(\frac{e}{4\pi R^3}\right)^2 \left(f^0_{\ \nu}\; f^{\mu\nu}_{\Phi} + f^0_{\ 5}\; f^{\mu 5}_{\Phi}\right) = \frac{1}{2}\left(\frac{e}{4\pi R^3}\right)^2 \left(\dot{u}R - uQ - \mathbf{e}_5 Q\right)^2 z^0 z^\mu$$

Instantaneously co-moving frame with on-shell velocity

$$u = (1, \mathbf{0}) \qquad u^2 = -1 \Rightarrow \dot{u}\cdot u = 0 \qquad \dot{u} = (0, \dot{\mathbf{v}})$$

$$z = (|\mathbf{z}|, \mathbf{z}) \qquad R = -u\cdot z = |\mathbf{z}| \qquad Q = -\dot{u}\cdot z = -\dot{\mathbf{v}}\cdot\mathbf{z}$$

Leading to

$$\left(\dot{u}R - uQ - \mathbf{e}_5 Q\right)^2 = \left[-\dot{u}\,(u\cdot z) + u\,(\dot{u}\cdot z) + \mathbf{e}_5\,(\dot{u}\cdot z)\right]^2 = \mathbf{z}^2\dot{\mathbf{v}}^2\left(\sin^2\theta - \cos^2\theta\right)$$

with contributions

$$\int d\tau \; T^{0\mu}_{spacetime} = \frac{1}{2}\left(\frac{e}{4\pi\,|\mathbf{z}|}\right)^2 \dot{\mathbf{v}}^2 \sin^2\theta\,(1, \hat{\mathbf{z}})$$

$$\int d\tau \; T^{0\mu}_{5} = -\frac{1}{2}\left(\frac{e}{4\pi\,|\mathbf{z}|}\right)^2 \dot{\mathbf{v}}^2 \cos^2\theta\,(1, \hat{\mathbf{z}})$$

Radiation reaction

Field along worldline

Action includes external field $a_\alpha^{in}(x,\tau)$ and induced field $a_\alpha(x,\tau)$

$$S_{em} = \int d\tau\ \frac{1}{2} m\dot{x}^2 + \int d^4x\, d\tau\ \lambda e j^\alpha(x,\tau) \left[a_\alpha(x,\tau) + a_\alpha^{in}(x,\tau)\right]$$
$$-\frac{\lambda}{4} \int d^4x\, d\tau \left[f^{\alpha\beta}(x,\tau) + f_{in}^{\alpha\beta}(x,\tau)\right] \left[f_{\alpha\beta}^{\Phi}(x,\tau) + f_{\alpha\beta}^{in\ \Phi}(x,\tau)\right]$$

Radiation field

$$f_{rad}^{\alpha\beta} = f_{out}^{\alpha\beta} - f_{in}^{\alpha\beta} = \left(f^{\alpha\beta} - f_{adv}^{\alpha\beta}\right) - \left(f^{\alpha\beta} - f_{ret}^{\alpha\beta}\right) = f_{ret}^{\alpha\beta} - f_{adv}^{\alpha\beta}$$

From Taylor expansion along worldline of

$$\partial^\mu a_{rad}^\beta(x,\tau) = -\frac{e}{2\pi} \int ds \left[\frac{d}{ds}\varphi(\tau - s) \frac{\dot{X}^\beta (x^\mu - X^\mu)}{\dot{X}\cdot(x - X)}\right] \delta\left[(x - X(s))^2\right] \epsilon\left(x^0 - X^0\right)$$

radiation field at event

$$f_{rad}^{\mu\nu}(X(s),\tau) = \varphi(\tau - s) \frac{2}{3}\frac{e}{2\pi} \left[u^\mu(s)\, \ddot{u}^\nu(s) - u^\nu(s)\, \ddot{u}^\mu(s)\right]$$

Radiation reaction

Lorentz force associated with radiation reaction

Force at point $x(\tau)$ including radiation field

$$
\begin{aligned}
m\ddot{x}^{\mu}(\tau) &= e_0 f_{in}^{\mu\nu}(x,\tau)\,\dot{x}_{\nu}(\tau) + e_0 f_{rad}^{\mu\nu}(x,\tau)\,\dot{x}_{\nu}(\tau) \\
&= e_0 f_{in}^{\mu\nu}(x,\tau)\,\dot{x}_{\nu} + \lambda\varphi(\tau-\tau)\,\frac{2}{3}\frac{e^2}{2\pi}\left[\dot{x}^{\mu}\ \dddot{x}^{\nu} - \dot{x}^{\nu}\ \dddot{x}^{\mu}\right]\dot{x}_{\nu} \\
&= e_0 f_{in}^{\mu\nu}(x,\tau)\,\dot{x}_{\nu} + \lambda\varphi(0)\,\frac{2}{3}\frac{e^2}{2\pi}\left[\dot{x}^{\mu}\ \dddot{x}^{\nu} - \dot{x}^{\nu}\ \dddot{x}^{\mu}\right]\dot{x}_{\nu} \\
&= e_0 f_{in}^{\mu\nu}(x,\tau)\,\dot{x}_{\nu} + \frac{2}{3}\frac{e^2}{4\pi}\left[\dot{x}^{\mu}\ \dddot{x}^{\nu}\ \dot{x}_{\nu} - \dot{x}^2\ \dddot{x}^{\mu}\right]
\end{aligned}
$$

On-shell event evolution

$$
u^2 = -1 \qquad u\cdot\dot{u} = 0 \qquad u\cdot\ddot{u} + \dot{u}^2 = 0
$$

Total Lorentz force

$$
m\ddot{x}^{\mu} = e_0 f_{in}^{\mu\nu}(x,\tau)\,\dot{x}_{\nu} + \frac{2}{3}\frac{e^2}{4\pi}\left[\dddot{x}^{\mu} - \ddot{x}^2\ \dot{x}^{\mu}\right]
$$

Integro-differential equation

Writing $\tau_0 = \frac{2}{3}\frac{e^2}{4\pi m}$

$$\ddot{x}^\mu(\tau) = \frac{e_0}{m} f_{in}^{\mu\nu}(x,\tau)\,\dot{x}_\nu + \tau_0\left[\dddot{x}^\mu - \ddot{x}^2\ \dot{x}^\mu\right]$$

$$\left(\ddot{x}^\mu - \tau_0\dddot{x}^\mu\right)e^{-\tau/\tau_0} = e^{-\tau/\tau_0}\left[\frac{e_0}{m} f_{in}^{\mu\nu}(x,\tau)\,\dot{x}_\nu - \tau_0\ddot{x}^2\ \dot{x}^\mu\right]$$

$$\ddot{x}^\mu e^{-\tau/\tau_0} = -\int_0^\tau ds\ e^{-s/\tau_0}\left[\frac{e_0}{\tau_0 m} f_{in}^{\mu\nu}(x,s)\,\dot{x}_\nu - \ddot{x}^2\ \dot{x}^\mu\right] + \ddot{x}^\mu(0)$$

If $\ddot{x}^\mu(\tau)\,e^{-\tau/\tau_0} \to 0$ as $\tau \to \infty$

$$0 = -\int_0^\infty ds\ e^{-s/\tau_0}\left[\frac{e_0}{\tau_0 m} f_{in}^{\mu\nu}(x,s)\,\dot{x}_\nu(s) - \ddot{x}^2(s)\,\dot{x}^\mu(s)\right] + \ddot{x}^\mu(0)$$

Incorporate $\ddot{x}^\mu(0)$

$$\ddot{x}^\mu(\tau) = e^{\tau/\tau_0}\int_\tau^\infty ds\ e^{-s/\tau_0}\left[\frac{e_0}{\tau_0 m}\ f_{in}^{\mu\nu}(x,s)\,\dot{x}_\nu(s) - \ddot{x}^2(s)\,\dot{x}^\mu(s)\right]$$

Change of variable

$$s \longrightarrow s' = s - \tau$$

ALD equation

$$\ddot{x}^{\mu}(\tau) = \int_0^{\infty} ds' \; e^{-s'/\tau_0} \left[\frac{e_0}{\tau_0 m} f_{in}^{\mu\nu}(x, s'+\tau)\, \dot{x}_{\nu}(s'+\tau) - \ddot{x}^2(s'+\tau)\, \dot{x}^{\mu}(s'+\tau) \right]$$

Geneology: field $f_{in}^{\mu\nu} \leftarrow$ smoothed current $j_{\varphi}(x,\tau) \leftarrow$ source event density $j(x,\tau)$

Write: field $\mathcal{F}_{in}^{\mu\nu} \leftarrow$ source event density $j(x,\tau)$

$$f_{in}^{\mu\nu}(x,\tau) = \int ds \; \varphi(\tau - s)\, \mathcal{F}_{in}^{\mu\nu}(x,s) = \int ds \frac{1}{2\lambda} e^{-|\tau - s|/\lambda} \; \mathcal{F}_{in}^{\mu\nu}(x,s)$$

For source event quantities

$$\ddot{x}^{\mu}(\tau) = \int_0^{\infty} ds' \; e^{-s'/\tau_0} \frac{e_0}{\tau_0 m} \int ds'' \frac{1}{2\lambda} e^{-|(s'+\tau-s'')|/\lambda} \; \mathcal{F}_{in}^{\mu\nu}(x,s'')\, \dot{x}_{\nu}(s'+\tau)$$
$$- \int_0^{\infty} ds' \; e^{-s'/\tau_0} \; \ddot{x}^2(s'+\tau)\, \dot{x}^{\mu}(s'+\tau)$$

Mediation by field relaxes event-event synchronization

External field and self-interaction exhibit similar statistical synchronization

Change order of integration

$$\int_0^\infty ds' \; e^{-s'/\tau_0} \tfrac{e_0}{\tau_0 m} \int ds'' \tfrac{1}{2\lambda} e^{-|(s'+\tau-s'')|/\lambda} \; \mathcal{F}_{in}^{\mu\nu}\left(x,s''\right) \dot{x}_\nu\left(s'+\tau\right)$$

$$= \int ds'' \tfrac{1}{2\lambda} e^{|\tau-s''|/\lambda} \; \mathcal{F}_{in}^{\mu\nu}\left(x,s''\right) \int_0^\infty ds' \; \tfrac{e_0}{m} \tfrac{1}{\tau_0} e^{-s'/\lambda'} \; \dot{x}_\nu\left(s'+\tau\right)$$

using $\lambda \simeq 4 \; \text{sec} \gg \tau_0 \simeq 6 \times 10^{-24} \; \text{sec} \;\Rightarrow\; 1/\lambda' = 1/\tau_0 + 1/\lambda \simeq 1/\tau_0$

ALD equation

$$\ddot{x}^\mu\left(\tau\right) = \lambda \frac{e}{m} \left\{ \mathcal{F}_{in}^{\mu\nu}\left(x,\tau\right) \right\}^\lambda \left\{ \dot{x}_\nu\left(\tau\right) \right\}^{\tau_0} - \tau_0 \left\{ \ddot{x}^2\left(\tau\right) \dot{x}^\mu\left(\tau\right) \right\}^{\tau_0}$$

where

$$\left\{ \mathcal{F}_{in}^{\mu\nu}\left(x,\tau\right) \right\}^\lambda = \int_{-\infty}^\infty ds \; \tfrac{1}{2\lambda} e^{-|\tau-s|/\lambda} \; \mathcal{F}_{in}^{\mu\nu}\left(x,s\right)$$

$$\left\{ \dot{x}_\nu\left(\tau\right) \right\}^{\tau_0} = \int_0^\infty ds \; \tfrac{1}{\tau_0} e^{-s/\tau_0} \; \dot{x}_\nu\left(s+\tau\right) = \int_{-\infty}^\tau ds \; \tfrac{1}{\tau_0} e^{-|\tau-s|/\tau_0} \; \dot{x}_\nu\left(2\tau-s\right)$$

$$\left\{ \ddot{x}^2\left(\tau\right) \dot{x}^\mu\left(\tau\right) \right\}^{\tau_0} = \int_{-\infty}^\tau ds \; \tfrac{1}{\tau_0} e^{-|\tau-s|/\tau_0} \; \ddot{x}^2\left(2\tau-s\right) \dot{x}^\mu\left(2\tau-s\right)$$