

# Duality in Off-Shell Electromagnetism

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## Abstract

In this paper, we examine the Dirac monopole in the framework of Off-Shell Electromagnetism, the five dimensional  $U(1)$  gauge theory associated with Stueckelberg-Schrodinger relativistic quantum theory. After reviewing the Dirac model in four dimensions, we show that the structure of the five dimensional theory prevents a natural generalization of the Dirac monopole, since the theory is not symmetric under duality transformations. It is shown that the duality symmetry can be restored by generalizing the electromagnetic field strength to an element of a Clifford algebra. Nevertheless, the generalized framework does not permit us to recover the phenomenological (or conventional) absence of magnetic monopoles.

## 1 Introduction

### 1.1 Duality

Dirac's 1931 model for the magnetic monopole [1], opened a number of interesting directions for research. Aside from providing a possible explanation for the quantization of electric charge, Dirac's work led to investigation of the topological aspects of gauge theory, and eventually, to the study of duality relations in non-Abelian and supersymmetric gauge theories. These areas of contemporary research emerged from Dirac's work because the seemingly simple question, 'Why is the magnetic field sourceless, while the electric field may be induced by a source?' touches on subtle aspects of the dimensionality of spacetime and the structure of admissible field theories. Although the question of the magnetic monopole can be approached by simply adding a magnetic current that makes Maxwell's equations symmetric under exchange of  $\mathbf{E}$  and  $\mathbf{H}$ , this symmetry relies on the duality relation between the two tensor equations,

$$\partial_\nu F^{\mu\nu} = J^\mu \quad (1)$$

$$\epsilon^{\mu\nu\lambda\rho} \partial_\nu F_{\lambda\rho} = \partial_\nu [\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}] = \partial_\nu \tilde{F}^{\mu\nu} = 0 \quad (2)$$

and the fact that  $\tilde{F}^{\mu\nu}$  is equivalent to  $F^{\mu\nu}$  with  $\mathbf{E} \rightarrow -\mathbf{H}$  and  $\mathbf{H} \rightarrow \mathbf{E}$ . The homogeneous Bianchi relation (2) connects the inhomogeneous equation (1) to the global gauge invariance associated with current conservation and to the local gauge invariance that leads to

potential theory. These deep features of the Maxwell theory provide the machinery that underly the symmetry under exchange of  $\mathbf{E}$  and  $\mathbf{H}$ , and the only ‘accidental’ element of this symmetry is that the theory is written in four dimensions where the Levi-Civita density has four indices, and hence the dual of the second rank field strength tensor is also a second rank tensor.

In this paper, we examine the question of duality in the framework of Off-Shell Electromagnetism [2], the five dimensional  $U(1)$  gauge theory associated with the Stueckelberg-Schrodinger relativistic quantum theory [3]. Although the five dimensions constitute a vector and a scalar representation of the four dimensional Lorentz group, the structure of the theory prevents a natural generalization of the Dirac monopole based on duality. We first show that the five dimensional generalization of the Maxwell equations do not possess duality symmetry. Subsequently, we demonstrate how the symmetry can be restored by generalizing the framework to a direct sum of tensors of multiple rank in a Clifford algebra. Finally, we show that although the duality symmetry is restored, the generalized framework does not permit the magnetic current to be ‘rotated away’ in order to recover the phenomenological (or conventional) absence of monopoles. We leave the topological aspects of the theory and the question of charge quantization to a subsequent paper.

## 1.2 Off-Shell Electromagnetism

Off-Shell Electromagnetism is a five dimensional generalization of the standard Maxwell theory, to which it reduces in an equilibrium limit. The field equations are

$$\partial_\beta f^{\alpha\beta}(x, \tau) = e j^\alpha(x, \tau) \qquad \epsilon^{\alpha\beta\gamma\delta\varepsilon} \partial_\alpha f_{\beta\gamma}(x, \tau) = 0 \qquad (3)$$

where  $\alpha, \beta, \gamma, \delta, \varepsilon = 0, 1, 2, 3, 5$ . The five indices correspond to the four dimensions of spacetime  $x^\mu$ , where  $\mu, \nu, \lambda = 0, 1, 2, 3$ , and a Poincaré invariant parameter  $x^5 = \tau$  that labels events along particle worldlines. The “fifth dimension” is formally similar to the Galilean invariant time in Newtonian theory, generalized to a covariant theory in which worldlines are traced out by the classical four-vector  $x^\mu(\tau)$  or the quantum wave function  $\psi(x, \tau)$ , as the parameter proceeds monotonically from  $\tau = -\infty$  to  $\tau = \infty$ . The fields act on particles through a covariant form of the classical Lorentz force originally proposed

by Stueckelberg [3],

$$\begin{aligned} M \ddot{x}_\mu &= \lambda e f_{\mu\alpha}(x, \tau) \dot{x}^\alpha = \lambda e \left[ f_{\mu\nu}(x, \tau) \dot{x}^\nu + f_{\mu 5}(x, \tau) \right] \\ \frac{d}{d\tau} \left( -\frac{1}{2} M \dot{x}^2 \right) &= \lambda e f_{5\alpha}(x, \tau) \dot{x}^\alpha = \lambda e f_{5\mu}(x, \tau) \dot{x}^\mu . \end{aligned} \quad (4)$$

This classical theory can be derived [4] from the quantum theory proposed by Sa'ad, Horwitz and Arshansky [2]

$$\left[ i\partial_\tau + \lambda e a_5(x, \tau) \right] \psi(x, \tau) = \frac{1}{2M} \left[ p^\mu - \lambda e a^\mu(x, \tau) \right] \left[ p_\mu - \lambda e a_\mu(x, \tau) \right] \psi(x, \tau) , \quad (5)$$

where the potentials are related to the field strengths through

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha . \quad (6)$$

The dimensional constant  $\lambda$  is required for consistency with Maxwell theory, as shown below. Equations (3) to (4) have been obtained [5] as the most general classical theory consistent with the commutations relations

$$[x^\mu, x^\nu] = 0 \quad m [x^\mu, \dot{x}^\nu] = -i\hbar g^{\mu\nu}(x) , \quad (7)$$

and as the dynamical theory associated with the invariance of (5) under local gauge transformations of the type

$$\psi(x, \tau) \rightarrow e^{i\lambda e \Lambda(x, \tau)} \psi(x, \tau) \quad a_\alpha(x, \tau) \rightarrow a_\alpha(x, \tau) + \partial_\alpha \Lambda(x, \tau) . \quad (8)$$

The inclusion of the parameter  $\tau$  in the gauge function [2] generalizes other parameterized quantum theories [3, 6], leading to a well-posed and integrable electrodynamics of interacting events. Relaxing the mass-shell constraint in (7) breaks general reparameterization invariance, but under the conditions

$$f_{5\mu} = 0 \quad \text{and} \quad \partial_\tau f^{\mu\nu} = 0 , \quad (9)$$

the remaining  $\tau$ -translation symmetry is associated, via Noether's theorem, with dynamic conservation of the mass. It has been shown [4] that while the material events and gauge fields may exchange mass when the conditions (9) do not hold, the total mass-energy of the particles and fields is conserved. Since the gauge fields propagate with a mass spectrum, this theory has been called off-shell electrodynamics. Equation (5) admits the five dimensional conserved current

$$\partial_\mu j^\mu + \partial_\tau j^5 = \partial_\mu \frac{-i}{2M} \left[ \psi^* (\partial^\mu - i\lambda e a^\mu) \psi - \psi (\partial^\mu + i\lambda e a^\mu) \psi^* \right] + \partial_\tau \left| \psi \right|^2 = 0 \quad (10)$$

leading to the interpretation of  $|\psi(x, \tau)|^2$  as the probability density at  $\tau$  of finding the event at the spacetime point  $x$ . The connection with Maxwell theory is made by extending an observation by Stueckelberg: if (9) holds asymptotically, pointwise in  $x$  as  $\tau \rightarrow \pm\infty$ , then integration of (3) over  $\tau$ , called concatenation of events into a worldlines [7], recovers standard Maxwell theory

$$\partial_\nu F^{\mu\nu} = eJ^\mu \quad \epsilon^{\mu\nu\rho\lambda} \partial_\mu F_{\nu\rho} = 0 \quad (11)$$

where

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau j^\mu(x, \tau) \quad \text{and} \quad F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau f^{\mu\nu}(x, \tau) \quad (12)$$

and so  $f^{\mu\nu}(x, \tau)$  has been called the pre-Maxwell field. It follows from (12) that  $\lambda$  has dimensions of length.

### 1.3 Spacetime Algebra

The spacetime algebra formalism [8] achieves a high degree of notational compactness by representing the usual tensorial objects of physics as intrinsic (basis independent, and hence index-free) elements in a Clifford algebra. Among the structural features of Clifford algebra that enable multiple physical statements to be combined are the direct sums of tensors of various rank and the natural separation of the product into a rank-lowering symmetric part and a rank-raising antisymmetric part. Furthermore, the close connection between Clifford numbers and geometry, has led to recent work [9] suggesting a generalization of standard physical entities through their representation as Clifford numbers. In this paper, we make use of both features of Clifford algebra, and here we introduce the most basic elements of the formalism.

Following Hestenes, we begin with the space of vectors  $a, b, c, \dots$  consisting of real-valued  $D$ -tuples

$$a = (a_0, a_1, \dots, a_{D-1}) \quad (13)$$

in Minkowski space with

$$g = \text{diag}(-1, 1, \dots, 1) \quad (14)$$

To construct the Clifford algebra  $\mathcal{C}_D$  we introduce a multiplication operation, with the requirement that Clifford numbers  $A, B, C, \dots$ , consisting of sums of products of any number of vectors, satisfy

1.  $A + B = B + A$
2.  $A + (B + C) = (A + B) + C$
3.  $A(BC) = (AB)C$
4.  $A(B + C) = AC + BC$
5.  $ab = a \cdot b$ , if and only if  $a$  and  $b$  are collinear.

From these requirements, one may derive [10] the rules of the spacetime algebra  $\mathcal{C}_D$ , a subset of which we give here without proof. The product of two vectors separates naturally into a symmetric part and antisymmetric part

$$ab = \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) = a \cdot b + a \wedge b \quad (15)$$

where the symmetric part can be identified with the scalar inner product, and the rank 2 antisymmetric part is called a bivector. The general Clifford number is a direct sum of multivectors of rank  $0, 1, \dots, D$

$$A = A_0 + A_1 + A_2 + A_3 + \dots + A_D \quad (16)$$

$$= A_0 + A_1^i \mathbf{e}_i + A_2^{ij} \mathbf{e}_i \wedge \mathbf{e}_j + \dots + A_D^{i_0 i_1 \dots i_{D-1}} \mathbf{e}_{i_0} \wedge \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_{D-1}} \quad (17)$$

expanded on the basis

$$\{1, \mathbf{e}_i, \mathbf{e}_i \wedge \mathbf{e}_j, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k, \dots, \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_{D-1}\} \quad (18)$$

The terms in (16) admit a geometric interpretation:  $A_1$  represents an oriented line,  $A_2$  represents an oriented plane,  $A_3$  represents an oriented volume, and so on. From the antisymmetry of the wedge product it follows that the  $r$ -product

$$\mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_r \quad (19)$$

in  $D$  dimensions spans a  $\binom{D}{r}$  dimensional subalgebra, and the dimension of the full Clifford algebra is

$$\dim \mathcal{C}_D = \sum_{r=0}^D \binom{D}{r} = 2^D \quad (20)$$

The product of a vector and an  $r$ -vector similarly separates into a symmetric part and antisymmetric part

$$aA_r = a(a_1 \wedge a_2 \wedge \dots \wedge a_r) = a \cdot A_r + a \wedge A_r \quad (21)$$

where

$$a \cdot A_r = \frac{1}{2} [aA_r - (-1)^r A_r a] \quad (22)$$

$$= \sum_{i=1}^r (-1)^{i+1} (a \cdot a_i) a_1 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \cdots \wedge a_r \quad (23)$$

$$a \wedge A_r = \frac{1}{2} [aA_r + (-1)^r A_r a] = a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r \quad . \quad (24)$$

The geometric interpretation of the separation in (21) is seen by taking a unit vector  $a^2 = 1$ . With a little algebra, it can be shown that

$$a \cdot [a (a \cdot A_r)] = a \cdot A_r = a \cdot A_r^{\parallel} \quad (25)$$

$$a \cdot [a (a \wedge A_r)] = 0 = a \cdot A_r^{\perp} \quad (26)$$

and it follows that the symmetric and antisymmetric parts of the product correspond to the linearly dependent and orthogonal components of the multivector  $A_r$

$$A_r = a^2 A_r = a (a \cdot A_r) + a (a \wedge A_r) = A_r^{\parallel} + A_r^{\perp} \quad . \quad (27)$$

In component form, linear independence of  $\mathbf{e}_k$  and  $A_r$  requires that  $A_r^{j_1 \cdots j_r} = 0$  when any of its indices takes the value  $k$ .

It follows from (20) that the  $D$ -vector in  $D$  dimensions spans a one-dimensional subalgebra, and so the element

$$i = \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{D-1} \quad (28)$$

is known as the unit pseudoscalar, and satisfies

$$i^2 = (-1)^{\frac{D(D-1)}{2}} g_{00} \cdots g_{D-1, D-1} \quad . \quad (29)$$

The unit pseudoscalar maps an  $r$ -vector to its dual multivector, a  $(D - r)$ -vector, through

$$i [\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r}] = g_{i_1 i_1} \cdots g_{i_r i_r} \frac{1}{(D - r)!} \varepsilon^{i_1 \cdots i_r i_{r+1} \cdots i_D} [\mathbf{e}_{i_{r+1}} \wedge \cdots \wedge \mathbf{e}_{i_D}] \quad (30)$$

providing the following useful relationships between the symmetric and antisymmetric products

$$a \cdot (iA_r) = \frac{1}{2} [aiA_r - (-1)^{D-r} (iA_r) a] \quad (31)$$

$$= \frac{1}{2} [(-1)^{D-1} iaA_r - (-1)^{D-r} (iA_r) a] \quad (32)$$

$$= (-1)^{D-1} \frac{1}{2} [iaA_r + (-1)^r iA_r a] \quad (33)$$

$$= (-1)^{D-1} i (a \wedge A_r) \quad (34)$$

and

$$a \wedge (iA_r) = \frac{1}{2} \left[ aiA_r + (-1)^{D-r} (iA_r) a \right] \quad (35)$$

$$= \frac{1}{2} \left[ (-1)^{D-1} iaA_r + (-1)^{D-r} (iA_r) a \right] \quad (36)$$

$$= (-1)^{D-1} \frac{1}{2} [iaA_r - (-1)^r iA_r a] \quad (37)$$

$$= (-1)^{D-1} i(a \cdot A_r) \quad (38)$$

In writing (31) it was necessary to use the known rank of  $(iA_r)$  in fixing the sign  $(-1)^{D-r}$ , and in the transition from (31) to (32) it was necessary to use the known rank and symmetry of  $ia$  in fixing the sign  $(-1)^{D-1}$ . The appearance of the common factor  $(-1)^D$  indicates the role of duality in these simple identities.

In analogy to complex numbers, the general Clifford number (16) can be written

$$A = A_0 + A_1 + A_2 + \cdots + A_{D-2} + A_{D-1} + A_D \quad (39)$$

$$= A_0 + A_1 + A_2 + \cdots + \tilde{A}_2 i + \tilde{A}_1 i + \tilde{A}_0 i \quad (40)$$

$$= (A_0 + \tilde{A}_0 i) + (A_1 + \tilde{A}_1 i) + (A_2 + \tilde{A}_2 i) + \cdots \quad (41)$$

where  $\tilde{A}_0$  is the scalar dual to  $A_D$ ,  $\tilde{A}_1$  is the vector dual to  $A_{D-1}$ ,  $\tilde{A}_2$  is the bivector dual to  $A_{D-2}$ , and so on.

## 2 Duality in Maxwell's Equations

### 2.1 Electrodynamics in Spacetime Algebra

In the spacetime algebra formalism, the Maxwell equations in  $D$ -dimensions assume the compact form

$$dF = J \quad (42)$$

As in the standard tensor formulation, the electromagnetic field strength is a bivector (antisymmetric second rank tensor), whose component form in four dimensions is

$$F = \frac{1}{2} F^{\alpha\beta} (\mathbf{e}_\alpha \wedge \mathbf{e}_\beta) \quad (43)$$

$$= [F^{01} (\mathbf{e}_0 \wedge \mathbf{e}_1) + F^{02} (\mathbf{e}_0 \wedge \mathbf{e}_2) + F^{03} (\mathbf{e}_0 \wedge \mathbf{e}_3) \\ + F^{12} (\mathbf{e}_1 \wedge \mathbf{e}_2) + F^{13} (\mathbf{e}_1 \wedge \mathbf{e}_3) + F^{23} (\mathbf{e}_2 \wedge \mathbf{e}_3)] \quad (44)$$

$$= [E^1 (\mathbf{e}_0 \wedge \mathbf{e}_1) + E^2 (\mathbf{e}_0 \wedge \mathbf{e}_2) + E^3 (\mathbf{e}_0 \wedge \mathbf{e}_3) \\ + H^3 (\mathbf{e}_1 \wedge \mathbf{e}_2) - H^2 (\mathbf{e}_1 \wedge \mathbf{e}_3) + H^1 (\mathbf{e}_2 \wedge \mathbf{e}_3)] \quad (45)$$

The gradient and current are written as vectors  $d = \partial^\alpha \mathbf{e}_\alpha$  and  $J = J^\alpha \mathbf{e}_\alpha$ . Equation (42) splits into the symmetric and antisymmetric parts

$$d \cdot F + d \wedge F = J \quad (46)$$

which can be separated by rank into

$$d \cdot F = J \quad (47)$$

$$d \wedge F = 0 \quad . \quad (48)$$

From (23) it follows that the components of (47) are the tensor form of the inhomogeneous Maxwell equations, which can be seen from

$$d \cdot F = \partial^\alpha \left( \frac{1}{2} F^{\beta\gamma} \right) \mathbf{e}_\alpha \cdot (\mathbf{e}_\beta \wedge \mathbf{e}_\gamma) = \partial^\alpha F^{\beta\gamma} \frac{1}{2} (g_{\alpha\beta} \mathbf{e}_\gamma - g_{\alpha\gamma} \mathbf{e}_\beta) = (\partial_\beta F^{\beta\gamma}) \mathbf{e}_\gamma \quad . \quad (49)$$

The total antisymmetry of  $\mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\gamma$  shows that (48) is the homogeneous Maxwell equation.

The dual to  $F$  is the multivector  $\tilde{F} = iF$ , given by

$$iF = \frac{1}{2} F^{\alpha\beta} (\mathbf{e}_0 \mathbf{e}_1 \dots \mathbf{e}_{D-1}) (\mathbf{e}_\alpha \wedge \mathbf{e}_\beta) = \frac{1}{2} \frac{1}{(D-2)!} \epsilon^{\beta\gamma\alpha_1\alpha_2\dots\alpha_{D-2}} F_{\beta\gamma} \mathbf{e}_{\alpha_1} \mathbf{e}_{\alpha_2} \dots \mathbf{e}_{\alpha_{D-2}} \quad (50)$$

so that in four dimensions, the dual is also a bivector

$$iF = \frac{1}{4} \epsilon^{\alpha_1\alpha_2\beta\gamma} F_{\beta\gamma} \mathbf{e}_{\alpha_1} \wedge \mathbf{e}_{\alpha_2} \quad (51)$$

$$= \left[ \epsilon^{0123} F_{01} \mathbf{e}_2 \wedge \mathbf{e}_3 + \epsilon^{0231} F_{02} \mathbf{e}_3 \wedge \mathbf{e}_1 + \epsilon^{0312} F_{03} \mathbf{e}_1 \wedge \mathbf{e}_2 \right. \\ \left. + \epsilon^{1230} F_{12} \mathbf{e}_3 \wedge \mathbf{e}_0 + \epsilon^{2310} F_{23} \mathbf{e}_1 \wedge \mathbf{e}_0 + \epsilon^{3120} F_{31} \mathbf{e}_2 \wedge \mathbf{e}_0 \right] \quad (52)$$

$$= \left[ E_1 \mathbf{e}_2 \wedge \mathbf{e}_3 - E_2 \mathbf{e}_1 \wedge \mathbf{e}_3 + E_3 \mathbf{e}_1 \wedge \mathbf{e}_2 \right. \\ \left. - H_3 \mathbf{e}_0 \wedge \mathbf{e}_3 - H_1 \mathbf{e}_0 \wedge \mathbf{e}_1 - H_2 \mathbf{e}_0 \wedge \mathbf{e}_2 \right] \quad . \quad (53)$$

Equation (53) is recognized as the field strength tensor (45) with the electric and magnetic fields exchanged. Using the relation (34), equation (48) can also be written

$$i(d \wedge F) = d \cdot (iF) = d \cdot \tilde{F} = 0 \quad . \quad (54)$$

## 2.2 Duality of the Electric and Magnetic Fields in $D = 4$

Following Dirac's famous argument, we may enhance the symmetry of Maxwell's equations by adding to (54) a non-zero current vector. Then, Maxwell's equations become

$$d \cdot F = J_{electric} \quad (55)$$

$$d \cdot \tilde{F} = J_{magnetic} \quad (56)$$

and in light of (54), (56) may be rewritten as a trivector relation

$$d \cdot \tilde{F} = d \cdot (iF) = i(d \wedge F) = J_{magnetic} \quad (57)$$

$$i^2(d \wedge F) = -(d \wedge F) = iJ_{magnetic} \quad (58)$$

$$d \wedge F = -iJ_{magnetic} \quad (59)$$

Combining (55) and (59) we find

$$d \cdot F + d \wedge F = dF = J_{electric} - iJ_{magnetic} = J \quad (60)$$

in which the current Clifford number  $J$  is now a direct sum of a vector and trivector. Equations (55) and (56) are form invariant under the exchange

$$F \leftrightarrow \tilde{F} \quad (61)$$

$$J_{electric} \leftrightarrow J_{magnetic} \quad (62)$$

and equivalently (60) is invariant under continuous duality transformations  $U(\theta) = e^{\theta i}$ , which by (29) can be written in  $D = 4$  as

$$U(\theta) = e^{\theta i} = \cos \theta + i \sin \theta \quad (63)$$

Applied to the field equations (60)

$$UdF = UJ \quad (64)$$

$$(1 + \theta i + o(\theta^2))(d \cdot F + d \wedge F) = (1 + \theta i + o(\theta^2))(J_{electric} - iJ_{magnetic}) \quad (65)$$

Separating terms of equal rank,

$$d \cdot F + \theta i d \wedge F = J_{electric} - \theta i^2 J_{magnetic} = J_{electric} + \theta J_{magnetic} = J'_{electric} \quad (66)$$

$$d \wedge F + \theta i d \cdot F = -i(J_{magnetic} - \theta J_{electric}) = -iJ'_{magnetic} \quad (67)$$

and using relations (34) and (38), we find form invariance in the form

$$d \cdot F - \theta d \cdot (iF) = d \cdot (F - \theta \tilde{F}) = d \cdot F' = J'_{electric} \quad (68)$$

$$d \wedge F - \theta d \wedge (iF) = d \wedge (F - \theta \tilde{F}) = d \wedge F' = -iJ'_{magnetic} \quad (69)$$

Dirac argued that the apparent absence of the magnetic current is actually a convention, according to which we view the world under the specific duality rotation which takes

$$J'_{magnetic} = (J_{magnetic}) \cos \theta - (J_{electric}) \sin \theta \rightarrow 0 \quad (70)$$

This argument requires (or implies) that the ratio of electric to magnetic charge  $e/g$  be universal. If the magnetic current exists, then it should contribute to the classical Lorentz force as

$$\mathbf{F} = e [\mathbf{E} + \mathbf{v} \times \mathbf{H}] + g [\mathbf{H} - \mathbf{v} \times \mathbf{E}] \quad . \quad (71)$$

Writing this Lorentz force in the covariant form

$$M \frac{d^2 x^i}{d\tau^2} = e F^{i\mu} \dot{x}_\mu + g \epsilon^{i\mu\nu\lambda} F_{\mu\nu} \dot{x}_\lambda = e F^{i\mu} \dot{x}_\mu + g \tilde{F}^{i\mu} \dot{x}_\mu \quad (72)$$

we see that under the duality rotation, this force becomes

$$M \frac{d^2 x^i}{d\tau^2} = e' (F')^{i\mu} \dot{x}_\mu \quad (73)$$

consistent with Dirac's argument that the absence of magnetic monopoles is a convention.

### 2.3 Duality of the Electric and Magnetic Fields in $D = 5$

The difficulties in extending Dirac's argument to five dimensions can be seen in a number of ways, perhaps most simply by observing that the 10 components of the electromagnetic field strength tensor do not divide naturally into a pair of vector fields  $\mathbf{E}$  and  $\mathbf{H}$  as occurs in  $D = 4$ . Similarly, the extension of the Lorentz force to five dimensions, in the form of (72), requires that the Levi-Cevita symbol be extended to five indices. A duality transformation leaving the expression

$$M \frac{d^2 x^i}{d\tau^2} = e F^{i\alpha} \dot{x}_\alpha + g \tilde{F}^{i\alpha} \dot{x}_\alpha = e F^{i\alpha} \dot{x}_\alpha + g \epsilon^{i\alpha\beta\gamma\delta} F_{\alpha\beta\gamma} \dot{x}_\delta \quad (74)$$

form invariant requires, at minimum the existence of a trivector field strength  $F_{\alpha\beta\gamma}$ . In this section, we demonstrate that, even in the spacetime algebra formalism which allows the representation of a physical object by a direct sum of various multivectors, Dirac's model for the monopole in U(1) electrodynamics cannot be reproduced when  $D \neq 4$ .

Summarizing the basic ingredients of Dirac's model, we require:

1. Maxwell's equations in the form  $dF = J = J_{(1)} + J_{(3)}$ , where the subscript labels the rank of the multivector,
2. A continuous transformation of the form

$$U = e^{\theta G} = 1 + \theta G + o(\theta^2) \quad (75)$$

generated by some Clifford number  $G$ ,

3. An algebraic structure in which the action of the transformation on Maxwell's equations

$$(1 + \theta G)(d \cdot F + d \wedge F) = (1 + \theta G)(J_{(1)} + J_{(3)}) \quad (76)$$

leads to separation of terms by rank,

$$d \cdot F + \theta G d \wedge F = J_{(1)} + \theta G J_{(3)} \quad (77)$$

$$d \wedge F + \theta G d \cdot F = J_{(3)} + \theta G J_{(1)} \quad (78)$$

4. A generator  $G$  which permits the replacements

$$G(d \wedge F) = d \cdot (GF) \quad (79)$$

$$G(d \cdot F) = d \wedge (GF) \quad (80)$$

so that the transformed multivectors can be recombined as

$$d \cdot F + \theta G d \wedge F = d \cdot (F + \theta GF) = J_{(1)} + \theta G J_{(3)} \quad (81)$$

$$d \wedge F + \theta G (d \cdot F) = d \wedge (F + \theta GF) = J_{(3)} + \theta G J_{(1)} \quad (82)$$

5. Form covariance expressed through the requirement that the pairs  $\{F, GF\}$  and  $\{J_{(3)}, GJ_{(1)}\}$  be of equal rank.

The problem in five dimensions begin in requirement (4), with finding an element  $G$  of  $\mathcal{C}_N$  which satisfies (79) and (80). Any choice of  $G$ , other than a scalar or pseudoscalar, determines an orthogonal subspace of vectors  $\{a_k \mid a_k \cdot G = 0\}$ , and it follows from (27) that the product  $GF$  splits into terms of various rank, according to

$$GF = GF^{\parallel} + GF^{\perp}$$

where  $a_k \cdot F^{\perp} = 0$ . The definite rank of the product  $iA_r$  was critical to the derivation of (34) and (38), and expressions (79) and (80) can only be satisfied when  $G = i$ . This choice implies that requirement (5) can only be satisfied in  $D = 4$ , where both  $F$  and  $iF$  are bivectors.

## 2.4 Generalized Field

It was seen in (74) that Dirac's model cannot be simply extended to five dimensions, because the duality transformation takes the bivector field into a trivector field. We

may ask whether the generalization of the electromagnetic field to include elements of higher rank can restore the symmetry under duality. In this section, we demonstrate that requirements (1) to (5) can be satisfied by writing the electromagnetic field tensor as a general Clifford number. We propose to represent the electromagnetic field and the source current as a direct sum of multivectors of all possible rank in  $D = 5$ ,

$$F = F_0 + F_1 + F_2 + F_3 + F_4 + F_5 \quad (83)$$

$$J = J_0 + J_1 + J_2 + J_3 + J_4 + J_5 \quad . \quad (84)$$

The left hand side of Maxwell's equations

$$dF = J \quad (85)$$

becomes

$$dF = dF_0 + dF_1 + dF_2 + dF_3 + dF_4 + dF_5 \quad (86)$$

$$\begin{aligned} &= dF_0 + d \cdot F_1 + d \wedge F_1 + d \cdot F_2 + d \wedge F_2 \\ &\quad + d \cdot F_3 + d \wedge F_3 + d \cdot F_4 + d \wedge F_4 + d \cdot F_5 \end{aligned} \quad (87)$$

where we used  $d \wedge F_D \equiv 0$ . Equating terms of equal rank, we find a system of five coupled equations,

$$d \cdot F_1 = J_0 \quad (88)$$

$$dF_0 + d \cdot F_2 = J_1 \quad (89)$$

$$d \wedge F_1 + d \cdot F_3 = J_2 \quad (90)$$

$$d \wedge F_2 + d \cdot F_4 = J_3 \quad (91)$$

$$d \wedge F_3 + d \cdot F_5 = J_4 \quad (92)$$

$$d \wedge F_4 = J_5 \quad . \quad (93)$$

Writing the duality transformation  $U(\theta) = e^{\theta i}$  for small  $\theta$  allows us to collect terms by rank on the right hand side,

$$J_r \rightarrow J_r + \theta i J_{5-r} \quad , \quad r = 0, \dots, 5 \quad (94)$$

and on the left hand side,

$$d \cdot F_1 \rightarrow d \cdot F_1 + \theta id \wedge F_4 \quad (95)$$

$$dF_0 + d \cdot F_2 \rightarrow dF_0 + d \cdot F_2 + \theta id \wedge F_3 + \theta id \cdot F_5 \quad (96)$$

$$d \wedge F_1 + d \cdot F_3 \rightarrow d \wedge F_1 + d \cdot F_3 + \theta id \wedge F_2 + \theta id \cdot F_4 \quad (97)$$

$$d \wedge F_2 + d \cdot F_4 \rightarrow d \wedge F_2 + d \cdot F_4 + \theta id \wedge F_1 + \theta id \cdot F_3 \quad (98)$$

$$d \wedge F_3 + d \cdot F_5 \rightarrow d \wedge F_3 + d \cdot F_5 + \theta id F_0 + \theta id \cdot F_2 \quad (99)$$

$$d \wedge F_4 \rightarrow d \wedge F_4 + \theta id \cdot F_1 \quad . \quad (100)$$

Applying (34) and (38), and collecting terms leads to the transformed equations

$$d \cdot (F_1 + \theta i F_4) = J_0 + \theta i J_5 \quad (101)$$

$$d(F_0 + \theta i F_5) + d \cdot (F_2 + \theta i F_3) = J_1 + \theta i J_4 \quad (102)$$

$$d \wedge (F_1 + \theta i F_4) + d \cdot (F_3 + \theta i F_2) = J_2 + \theta i J_3 \quad (103)$$

$$d \wedge (F_2 + \theta i F_3) + d \cdot (F_4 + \theta i F_1) = J_3 + \theta i J_2 \quad (104)$$

$$d \wedge (F_3 + \theta i F_2) + d \cdot (F_5 + \theta i F_0) = J_4 + \theta i J_1 \quad (105)$$

$$d \wedge (F_4 + \theta i F_1) = J_5 + \theta i J_0 \quad . \quad (106)$$

Following Dirac, we may identify the transformed equations (101) to (106) with the Maxwell equations in some dual system

$$d \cdot F'_1 = J'_0 \quad (107)$$

$$dF'_0 + d \cdot F'_2 = J'_1 \quad (108)$$

$$d \wedge F'_1 + d \cdot F'_3 = J'_2 \quad (109)$$

$$d \wedge F'_2 + d \cdot F'_4 = J'_3 \quad (110)$$

$$d \wedge F'_3 + d \cdot F'_5 = J'_4 \quad (111)$$

$$d \wedge F'_4 = J'_5 \quad , \quad (112)$$

and comparing (107) — (112) with (88) — (93), it is clear that the duality symmetry has been restored, and that this procedure can be repeated in any number of dimensions.

However, we cannot claim that in general, the duality transformation that eliminates the magnetic current  $J'_3$  will eliminate all of the non-vector currents. To see this, we rewrite (90), (92) and (93) in a dual form, by defining

$$H_r = iF_{5-r} \quad , \quad r = 1, 3, 4, 5 \quad (113)$$

$$Q_s = iJ_{5-s} \quad , \quad s = 2, 4, 5 \quad (114)$$

so that

$$i[d \wedge F_1 + d \cdot F_3 = J_2] \rightarrow d \cdot (iF_1) + d \wedge (iF_3) = (iJ_2) \quad (115)$$

$$i[d \wedge F_3 + d \cdot F_5 = J_4] \rightarrow d \cdot (iF_3) + d \wedge (iF_5) = (iJ_4) \quad (116)$$

$$i[d \wedge F_4 = J_5] \rightarrow d \cdot (iF_4) = (iJ_5) \quad . \quad (117)$$

become

$$d \cdot H_4 + d \wedge H_2 = Q_3 \quad (118)$$

$$d \cdot H_2 + dH_0 = Q_1 \quad (119)$$

$$d \cdot H_1 = Q_0 \quad . \quad (120)$$

Replacing (90), (92) and (93) with (118), (119) and (120), the six field equations can be written as

$$d \cdot F_1 = J_0 \quad (121)$$

$$dF_0 + d \cdot F_2 = J_1 \quad (122)$$

$$d \wedge F_2 + d \cdot F_4 = J_3 \quad (123)$$

$$d \cdot H_1 = Q_0 \quad (124)$$

$$dH_0 + d \cdot H_2 = Q_1 \quad (125)$$

$$d \wedge H_2 + d \cdot H_4 = Q_3 \quad , \quad (126)$$

so that we are led to two inequivalent sets of coupled field equations of rank 0, 1, and 3. There is no guarantee that the angle  $\theta$  that transforms  $J_3 \rightarrow J'_3 = 0$  will also take  $Q_3 \rightarrow Q'_3 = 0$ . Thus, extending the electromagnetic field to a general Clifford number does permit the restoration of duality symmetry, but does not permit the elimination of the magnetic current.

### 3 Conclusion

We have seen that in five dimensions, the usual treatment of electromagnetic duality cannot be constructed. The duality symmetry which exists in four dimensions can be minimally restored by generalizing the electromagnetic field to the Clifford algebra form

$$F = F_2 + F_3 \quad (127)$$

$$J = J_1 + J_2 + J_3 + J_4 \quad , \quad (128)$$

in which Maxwell's equations

$$dF = J \quad (129)$$

split into

$$d \cdot F_2 = J_1 \quad (\text{vector}) \quad (130)$$

$$d \wedge F_2 = J_3 \quad (\text{trivector}) \quad (131)$$

$$d \cdot F_3 = J_2 \quad (\text{bivector}) \quad (132)$$

$$d \wedge F_3 = J_4 \quad (4 - \text{vector}) \quad (133)$$

and the duality transformation  $U(\theta) = e^{\theta i}$  balances (130) with (133) and (131) with (132) through

$$id \cdot F_2 = iJ_1 \rightarrow d \wedge (iF_2) = (iJ_1) \quad (4 - \text{vector}) \quad (134)$$

$$id \wedge F_2 = iJ_3 \rightarrow d \cdot (iF_2) = (iJ_3) \quad (\text{bivector}) \quad (135)$$

$$id \cdot F_3 = iJ_2 \rightarrow d \wedge (iF_3) = (iJ_2) \quad (\text{trivector}) \quad (136)$$

$$id \wedge F_3 = iJ_4 \rightarrow d \cdot (iF_3) = (iJ_4) \quad (\text{vector}) \quad (137)$$

However, this same duality symmetry permits, through the definitions

$$H_2 = iF_3 \quad (138)$$

$$Q_1 = iJ_4 \quad (139)$$

$$Q_3 = iJ_2 \quad , \quad (140)$$

the identification of (132) and (133) as a second set of Maxwell equations

$$d \cdot H_2 = Q_1 \quad (141)$$

$$d \wedge H_2 = Q_3 \quad , \quad (142)$$

and in general, the choice of  $\theta$  which eliminates  $J_3$  will not also eliminate  $Q_3$ . Thus, while the duality symmetry has been restored, it cannot be exploited to explain the absence of magnetic monopole currents.

The application of Dirac's argument to off-shell electromagnetism requires that the duality transformation not include the fifth component. The transformation remains, as in four dimensions,  $\mathbf{E} \rightarrow \mathbf{H}$  and  $\mathbf{H} \rightarrow -\mathbf{E}$ . In the spacetime algebra formalism, this transformation is expressed using

$$i_4 = \mathbf{e}_0 \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3 \quad (143)$$

which is the unit pseudoscalar in  $D = 4$ , and in five dimensions is orthogonal to the unit vector in the 5-direction  $\mathbf{e}_5$ . Since the choice of orientation which distinguishes the 5-direction is natural to the off-shell theory, covariance is preserved. It follows from (53) that the transformation which swaps the electric and magnetic fields is

$$F' = i_4 (F^\perp) + F^\parallel \quad (144)$$

$$= i_4 [\mathbf{e}_5 (\mathbf{e}_5 \wedge F)] + [\mathbf{e}_5 (\mathbf{e}_5 \cdot F)] \quad (145)$$

$$= i_4 \frac{1}{2} F^{\mu\nu} (\mathbf{e}_\mu \wedge \mathbf{e}_\nu) + F^{\mu 5} (\mathbf{e}_\mu \wedge \mathbf{e}_5) \quad (146)$$

where  $\mu, \nu = 0, \dots, 3$ . It is interesting to note that since this transformation leaves the 5-direction unchanged, Dirac's argument for the absence of magnetic monopole currents requires that  $J_{magnetic}^5 \equiv 0$ , *a priori*, so that it is distinguished from  $J_{electric}^5$  by the form of current conservation

$$\partial_\alpha J_{electric}^\alpha = \partial_\tau J_{electric}^5 + \partial_\mu J_{electric}^\mu = 0, \quad \alpha = 0, \dots, 3, 5 \quad (147)$$

$$\partial_\mu J_{magnetic}^\mu = 0, \quad \mu = 0, \dots, 3 \quad . \quad (148)$$

While  $J_{electric}^\alpha$  is understood, in analogy to the classical non-relativistic case, to be generated by a  $\tau$ -dependent event density  $J_{electric}^5$  conserved through its flow  $J_{electric}^\mu$  into spacetime, the magnetic current appears to be pure spacetime current circulation, not associated with any endpoint event density. In this sense, the magnetic current is again distinguished from the electric current by topological considerations, which we consider in a subsequent paper.

## References

- [1] P. A. M. Dirac, *Proc. R. Soc. A* **113** (1931) 60.
- [2] D. Saad, L. P. Horwitz and R. I. Arshansky, *Found. of Phys.* **19** (1989) 1126.
- [3] E. C. G. Stueckelberg, *Helv. Phys. Acta* **14** (1941) 322,  
E. C. G. Stueckelberg, *Helv. Phys. Acta* **14** (1941) 588.
- [4] M. C. Land and L. P. Horwitz, *Found. of Phys. Lett.* **4** (1991) 61.
- [5] M. C. Land, N. Shnerb, L. P. Horwitz, *J. Math. Phys.* **36** (1995) 3263.

- [6] V. A. Fock, *Phys. Z. Sowjetunion* **12** (1937) 404,  
R. P. Feynman, *Phys. Rev.* **80** (1950) 440,  
J. Schwinger, *Phys. Rev.* **82** (1951) 664,  
Y. Nambu, *Prog. Theor. Phys.* **5** (1950) 82,  
L. P. Horwitz and C. Piron, *Helv. Phys. Acta* **48** (1973) 316,  
R. Fanchi, *Phys. Rev. D* **20** (1979) 3108,  
A. Kyprianides, *Phys. Rep.* **155** (1986) 1.
- [7] R. Arshansky, L. P. Horwitz and Y. Lavie, *Found. of Phys.* **13** (1983) 1167.
- [8] David Hestenes, *Space-Time Algebra*, Gordon and Breach, New York, 1966.
- [9] Matej Pavsic, *The Landscape of Theoretical Physics: A Global View*, Kluwer Academic Publishers, Dordrecht, 2001,  
T. Ivezić, *The Proof that the Standard Transformations of  $E$  and  $B$  are not the Lorentz Transformations. Clifford Algebra Formalism*,  
<http://arxiv.org/physics/0304085>.
- [10] M. C. Land, *The Clifford Algebra Over  $R^n$* , unpublished undergraduate thesis, 1977.