

The Lorentz Force and Energy-Momentum for Off-Shell Electromagnetism

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Abstract: The kinematics of pre-Maxwell electrodynamics is examined and interpretations of these fields is found through an examination of the associated Lorentz force and the structure of the energy-momentum tensor.

Key words: relativistic quantum theory, off-shell gauge fields, pre-Maxwell fields, higher dimensional gauge fields.

In the framework of a covariant relativistic quantum theory [1][2], in which the evolution of the system is parametrized by a universal $O(3,1)$ invariant world (or historical) time τ , Saad, Horwitz and Arshansky [3] have shown that the requirement of local gauge invariance leads to five compensation fields. These fields, which have been called pre-Maxwell fields, are defined on a five dimensional manifold with coordinates (x^μ, τ) , and the gauge symmetry of the equations of motion is associated with a five component conserved current consisting of an $O(3,1)$ four-vector current, j^μ , and a scalar density (in R^4), ρ . The field equations, obtained from a Lagrangian constructed from gauge invariant field strengths, have a five dimensional symmetry which could be $O(3,2)$ or $O(4,1)$. The propagators for these field equations, classified according to their spacetime asymptotic properties, have been worked out in ref. [4].

In this letter we shall examine the kinematics of the pre-Maxwell electrodynamics and find interpretations of the new fields through an examination of the Lorentz force and the structure of the energy-momentum tensor of the field.

The evolution equation [1] (we use the metric $-+++$ for the $O(3,1)$ indices)

$$i\frac{\partial}{\partial\tau}\psi_\tau = \frac{p^\mu p_\mu}{2M}\psi_\tau \quad (1)$$

may be made locally gauge invariant under the transformations (we shall use $x \equiv x^\mu$)

$$\psi \rightarrow e^{ie_0\Lambda(x,\tau)}\psi \quad (2)$$

through the introduction of compensation fields $a_\alpha = (a_\mu, a_4)$ which transform as ($\partial_\alpha = \partial/\partial x^\alpha, dx^4 = d\tau$)

$$a_\alpha(x, \tau) \rightarrow a_\alpha(x, \tau) + \partial_\alpha\Lambda(x, \tau) \quad (3)$$

The gauge invariant evolution equation then becomes

$$i\frac{\partial}{\partial\tau}\psi_\tau(x, \tau) = \left[\frac{1}{2M}(p_\mu - e_0a_\mu)(p^\mu - e_0a^\mu) - e_0a_4\right]\psi_\tau(x). \quad (4)$$

It follows from this equation (as for the case of τ -independent Maxwell fields treated by Stueckelberg [1]) that

$$\partial_\mu j^\mu + \partial_\tau \rho = 0 \quad (5)$$

i.e.,

$$\partial_\alpha j^\alpha = 0 \quad (6)$$

where

$$j_\tau^4 = \rho = |\psi_\tau(x)|^2$$

and

$$j_\tau^\mu = -\frac{i}{2M}[\psi_\tau^*(\partial_\mu - ie_0a^\mu)\psi_\tau - \psi_\tau(\partial_\mu + ie_0a^\mu)\psi_\tau^*] \quad (7)$$

We shall, in the following, consider the right hand side of (4) as providing the form of the classical relativistic generator of τ -evolution, i.e., the Hamiltonian K for the classical mechanics of events in spacetime. The canonical Hamilton equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} = \frac{p^\mu - e_0a^\mu}{M}$$

enable us to write the Lagrangian (we shall write four vector products as e.g., $p^2 \equiv p^\mu p_\mu, x \cdot p \equiv x^\mu p_\mu$)

$$L = \frac{dx}{d\tau} \cdot p - K$$

as

$$L = \frac{M}{2}\left(\frac{dx}{d\tau}\right)^2 + e_0\left(a_\alpha \frac{dx^\alpha}{d\tau}\right), \quad (8)$$

where we have used the fact that $dx^4/d\tau = 1$. To provide the pre-Maxwell fields with dynamical structure, we introduce a kinetic term for the field to the Lagrangian. The total action is then

$$S = \int d\tau \frac{M}{2}\left(\frac{dx}{d\tau}\right)^2 + e_0 a_\alpha \frac{dx^\alpha}{d\tau} - \frac{\lambda}{4} \int d\tau d^4x' f^{\alpha\beta}(x', \tau) f_{\alpha\beta}(x', \tau), \quad (9)$$

where

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \quad (10)$$

To raise and lower the fifth index $\alpha = 4$, we use a formal metric signature that we shall call $\sigma = \pm$ (to be specified later), i.e., our five dimensional signature is of the form $(-, +, +, +, \sigma)$. The first, $O(3,1)$ invariant, term of the Lagrangian, associated with the matter equations, breaks the $O(3,2)$ or $O(4,1)$ symmetry corresponding to this metric.

The variation of this action with respect to the path $x(\tau)$ leads to the equation of motion

$$M \frac{d^2 x^\mu}{d\tau^2} = e_0 f^{\mu\alpha} \frac{dx_\alpha}{d\tau} = e_0 (f^{\mu\nu} \frac{dx_\nu}{d\tau} - \sigma f^{4\mu}), \quad (11)$$

the Lorentz force associated with the pre-Maxwell fields.

As for the usual Maxwell analog, we calculate the derivative of the kinetic term for the event motion:

$$\frac{d}{d\tau} \left[\frac{1}{2} M \left(\frac{dx}{d\tau} \right)^2 \right] = M \left(\frac{dx}{d\tau} \right) \cdot \left(\frac{d^2 x}{d\tau^2} \right) = -\sigma e_0 f^{4\mu} \frac{dx_\mu}{d\tau}, \quad (12)$$

where we have used (11) and the antisymmetry of $f^{\mu\nu}$. Hence we see with the help of (12) that one obtains the concise form

$$M \frac{d}{d\tau} (p^\alpha - e_0 a^\alpha) = e_0 f^{\alpha\beta} \frac{dx_\beta}{d\tau} \quad (13)$$

for a formal ‘‘five vector’’ $p_\alpha = (p_\mu, K_{kin})$, where K_{kin} is the kinetic part of K . This result is in the form of a generalized Lorentz force which is formally covariant over a larger group with the signature $(-, +, +, +, \sigma)$, i.e., $O(3,2)$ or $O(4,1)$. We note that the Lorentz force is proportional to e_0 , not the dimensionless Maxwell charge e (the field strengths $f^{\alpha\beta}$ have dimension $\frac{1}{L^4}$).

We remark that the integral of (12) over τ results in

$$\frac{1}{2M} (m^2(\infty) - m^2(-\infty)) = \sigma e_0 \int_{-\infty}^{\infty} d\tau f^{4\mu} \left(\frac{dx_\mu}{d\tau} \right). \quad (14)$$

where $m^2(x) = -(p - e_0 a)^2$. The four-vector $f^{4\mu}$ is seen here to generate changes in the Lorentz invariant mass squared. Asymptotic conservation of mass would imply a systematic cancellation on the right hand side side, for example, through oscillations in $dx^\mu/d\tau$ or for $f^{4\mu}$ spacelike at $x^\mu(\tau)$ for each τ . The mass squared may, however, undergo local dynamically induced variations even if it is asymptotically conserved.

The phenomenon of (classical) pair annihilation or production [1] requires such a mass change; since $K = -(M/2)(ds^2/d\tau^2) - e_0 a_4$, a change in sign of $ds^2 = dt^2 - d\mathbf{x}^2$ may occur when compensated by a_4 . In the Maxwell analog of the extension of (11) to (13), the derivative of the usual kinetic energy is proportional to $F^{0j} v_j (= \mathbf{E} \cdot \mathbf{v})$, where $v_j = dx_j/dt$, and $F^{\mu\nu}$ is the Maxwell field, and vanishes when \mathbf{v} does. Both for relativistic and nonrelativistic kinematics, vanishing of the kinetic energy implies that $\mathbf{v} = 0$, and hence no further decrease (to negative values, corresponding to a turning back of the trajectory) is possible. In Eq. (12), it is clear that vanishing of the *mass* (squared), i.e., $(\frac{dx}{d\tau})^2$, does not imply that $dx^\mu/d\tau$ vanishes. Hence, it is possible for the motion to pass through the light cone, resulting in possible pair annihilation. Stueckelberg [5] found that

although classical pair creation and annihilation can, in principle, occur in the framework of the manifestly covariant theory, such processes are not induced by standard Maxwell fields (the sign of the invariant interval is conserved). We see, in the analysis of (11) and (12), that τ -dependence of the fields is essential (so that $f^{4\mu}$ is not zero). We remark that, in the pre-Maxwell theory, moreover, if $dx^\mu/d\tau = 0$, the $\alpha = 4$ term contributes to $d^2x^\mu/d\tau^2$ the quantity $e_0 f^{\mu 4}$; hence the sign of $dx^0/d\tau$ can change under the influence of the $f^{\mu 4}$ field. An example of this phenomenon is given in the next section.

The variation of the action with respect to the fields $a_\alpha(x, \tau)$ leads to the field equations

$$\partial_\beta f^{\alpha\beta}(x, \tau) = (e_0/\lambda) \left(\frac{d}{d\tau} x^\alpha(\tau) \right) \delta^4(x - x(\tau)) \equiv e j^\alpha(x, \tau) \quad (15)$$

where $e \equiv e_0/\lambda$. Integrating the $\alpha = \mu$ components of this equation on $(-\infty, \infty)$ over τ , with the condition that $f^{\mu 4}$ vanishes pointwise at $\tau \rightarrow \pm\infty$, we recover the Maxwell equation [3][4]

$$\partial_\nu F^{\mu\nu}(x) = e J^\mu(x) \quad (16)$$

It follows, as for the quantum case (5), from (this condition is necessary for the consistency of (15) and the gauge invariance of the action) [1][2]

$$\partial_\mu j^\mu(x, \tau) + \frac{\partial \rho}{\partial \tau} = \partial_\alpha j^\alpha(x, \tau) = 0 \quad (17)$$

where $\rho = \delta^4(x - x(\tau))$, that

$$J^\mu(x) = \int d\tau j^\mu(x, \tau) \quad (18)$$

is conserved if the density ρ vanishes pointwise at $\tau \rightarrow \pm\infty$. This integration for J^μ has been called ‘‘concatenation’’ [6], and provides the link between the notion of an *event* along a world line and the notion of a *particle*, whose support in spacetime is the whole world line. We see from (16) that the Maxwell potential is given by the concatenation of the pre-Maxwell field

$$A^\mu(x) = \int d\tau a^\mu(x, \tau) \quad (19)$$

and that the constants e_0 and λ have the dimension of length. Their ratio is identified by (16) as the dimensionless electric charge. The representation of $f_{\alpha\beta}$ as the antisymmetric derivative of a five- field is equivalent to the homogeneous equation

$$\partial^\alpha \epsilon_{\alpha\beta\gamma\delta\sigma} f^{\delta\sigma} = 0, \quad (20)$$

analogous to the homogeneous Maxwell equations.

We now examine the noncovariant form of the field equations [3]. Let us define the three-vectors

$$e_i = f^{0i} \quad h_i = \frac{1}{2} \epsilon_{ijk} f^{jk}, \quad (21)$$

and the four-vector ϵ^μ defined by

$$\epsilon^i = f^{4i} \quad \epsilon^0 = f^{40}. \quad (22)$$

The pre-Maxwell equations then become

$$\begin{aligned}
\nabla \cdot \mathbf{e} &= ej^0 + \partial_\tau \epsilon^0 & \nabla \times \mathbf{e} + \partial_0 \mathbf{h} &= 0 \\
\nabla \times \mathbf{h} - \partial_0 \mathbf{e} - \partial_\tau \epsilon &= e\mathbf{j} & \nabla \cdot \mathbf{h} &= 0 \\
\nabla \cdot \epsilon &= ej^4 - \partial_0 \epsilon^0 & \nabla \times \epsilon - \sigma \partial_\tau \mathbf{h} &= 0 \\
\nabla \epsilon^0 &= -\sigma \partial_\tau \mathbf{e} - \partial_0 \epsilon & &
\end{aligned} \tag{23}$$

In the static limit, for which there is no τ -dependence in the fields, this system of equations reduces to Maxwell's form for the fields \mathbf{e} and \mathbf{h} ; the conditions on ϵ become

$$\begin{aligned}
\nabla \times \epsilon &= 0 \\
\nabla \epsilon^0 + \partial_0 \epsilon &= 0 \\
\partial_\mu \epsilon^\mu &= ej^4,
\end{aligned} \tag{24}$$

where the last of (24) is generally valid. From the first of (24), it follows that there exists a function $\phi(x)$ such that

$$\epsilon = -\nabla \phi, \tag{25}$$

and from the second that $\nabla(\epsilon^0 + \partial^0 \phi) = 0$. We therefore may take (up to a constant in ϵ^0)

$$\epsilon^\mu = -\partial^\mu \phi. \tag{26}$$

The third of (24) then implies that

$$-\partial^\mu \partial_\mu \phi = ej^4 = e\rho. \tag{27}$$

The field ϵ^μ is therefore defined by a scalar potential in the static case; the potential is a Klein-Gordon type massless field whose source is the (static; e.g., $\delta^4(x)$) event density. It is decoupled from the Maxwell type fields. The field equations, as for the Maxwell theory, imply the existence of the vector potential a^μ , for which (10) is valid [3]. Since

$$-\partial^\mu \phi = \epsilon^\mu = f^{4\mu} = -\partial^\mu a^4, \tag{28}$$

in the static case, we see that (26) is equivalent to (28) when ϕ is identified with a^4 , the additive potential term in the generator of evolution K . As in the nonrelativistic case, we find that the scalar (in this case Lorentz scalar) component of the gauge field plays the role of a potential in the static limit. Note that a_0 does not change sign under charge conjugation, but the a_4 field does.

We now study the noncovariant form of the Lorentz force. The space components of (11) have the form

$$M\ddot{\mathbf{x}} = e_0[\mathbf{e}\dot{x}^0 + \dot{\mathbf{x}} \times \mathbf{h} - \sigma\epsilon] \tag{29}$$

where $\dot{x}^\mu = dx^\mu/d\tau$. Since $\dot{x}^0 = (m/M)\frac{1}{\sqrt{1-v^2}} = (m/M)\gamma$ (for $|\mathbf{v}| = |d\mathbf{x}/dt| < 1$), we may write this equation in the form

$$\frac{d}{dt}m\gamma\mathbf{v} = e_0(\mathbf{e} + \mathbf{v} \times \mathbf{h} - \sigma\epsilon(M/m\gamma)) \quad (30)$$

This equation has the form of the usual Lorentz force, with the additional contribution of the ϵ field, which becomes less significant as γ becomes large. Since the left hand side of (30) has a classical interpretation in terms of motion along the world line, we see that $e_0(\mathbf{e} - \sigma M\epsilon/m\gamma)$ is an effective “electric” force. The 0th component of (11),

$$M\ddot{x}^0 = e_0(\mathbf{e} \cdot \dot{\mathbf{x}} - \sigma\epsilon^0), \quad (31)$$

can be written as

$$\frac{d}{dt}(\gamma m) = e_0(\mathbf{e} \cdot \mathbf{v} - \sigma M\epsilon^0/m\gamma); \quad (32)$$

it represents the work performed on the charge by the field, and is associated with a change in the “energy”. The noncovariant form for the 4 component of (12) is

$$\frac{d}{dt}\frac{m^2}{2M} = \sigma e_0(\epsilon \cdot \mathbf{v} - \epsilon^0); \quad (33)$$

where we have replaced $(dx/d\tau)^2$ by $-(m^2/M^2)$. We remark that the magnetic field \mathbf{h} does not contribute to the change in γm or the kinetic term $(m^2/2M)$. Combining (32) and (33) to eliminate the ϵ^0 term, one obtains

$$\frac{d}{dt}(\gamma m) - \frac{1}{\gamma}\frac{dm}{dt} = e_0\left[\mathbf{e} - \frac{\sigma M}{\gamma m}\epsilon\right] \cdot \mathbf{v}, \quad (34)$$

the linear combination of \mathbf{e} and ϵ which, as pointed out above, corresponds to an effective “electric” force. Taking the scalar product of (30) with \mathbf{v} , the term $\mathbf{v} \times \mathbf{h}$ drops out, and we find that $d/dt(\gamma m) - \frac{1}{\gamma}dm/dt = \mathbf{v} \cdot d/dt(m\gamma\mathbf{v})$; the term proportional to dm/dt cancels in this expression, which then reduces to an identity. This relation reflects the fact that eq. (33) is not independent of (30) and (32). We also note that in the case $dm/dt = 0$, it follows from (33) that $\epsilon^0 = \epsilon \cdot \mathbf{v}$; (32) then coincides with (34).

Let us consider an example in which the world line changes the direction of its evolution in t due to the presence of the force $-\sigma e_0\epsilon^0$. The event enters a region at $\tau=0$ in which $f^{\mu\nu} = f^{4i} = 0$, and $\epsilon^0 = \sigma|\epsilon^0|$, with ϵ^0 constant, (31) becomes

$$M\ddot{x}^0 = e_0|\epsilon^0|, \quad (35)$$

and hence, the $dt/d\tau$ changes sign when τ becomes sufficiently large. Note that this effect is independent of velocity. Stueckelberg [5] found, in his study, that it was necessary to add another, *ad hoc*, vector field in order to achieve pair creation. In fact, the structure of the Maxwell interaction assures that the square of the proper time, as we have pointed

out above, is conserved; as seen from (12), $f^{4\mu}$ provides the possibility of a reversal in the direction of the world line.

As an another example of how the Lorentz force for the pre-Maxwell case differs from that for the Maxwell case, consider motion in a constant electric field along the z -axis. The Lorentz force for the Maxwell fields has the form

$$m \frac{d^2}{ds^2} x^\mu = e F^{\mu\nu} \frac{dx_\nu}{ds} \quad (36)$$

which in the case $\mathbf{E} = E\hat{\mathbf{z}}$ becomes

$$m \frac{d}{ds} \begin{pmatrix} \dot{t} \\ \dot{z} \end{pmatrix} = e \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \begin{pmatrix} \dot{t} \\ \dot{z} \end{pmatrix} \quad (37)$$

Taking boundary conditions, $\dot{z}(0) = \frac{dz(0)}{ds} = 0$; $\dot{t}(0) = 1$; $t(0) = 0$, this first order differential equation has the solution

$$t(s) = \frac{m}{eE} \sinh \frac{eE}{m} s \quad z(s) = \frac{m}{eE} (\cosh \frac{eE}{m} s - 1) + z(0) \quad (38)$$

which can be rewritten as

$$z(t) = z(0) + \frac{m}{eE} (\sqrt{1 + (eEt/m)^2} - 1) \quad (39)$$

where we have used $dt/ds = 1/\sqrt{1 - (dz/dt)^2}$ and $\cosh[\sinh^{-1}(x)] = \sqrt{1 + x^2}$.

The pre-Maxwell equivalent of such a constant (in space-time) electric field is given by $\mathbf{e} = e(\tau)\hat{\mathbf{z}}$, where $\int d\tau e(\tau) = E$. However, from the last of eqs. (23), it is evident that a τ -dependent \mathbf{e} must be accompanied by a space-time-dependent ϵ field. We may control this complication by deriving the pre-Maxwell field from

$$a^3(x, \tau) = \begin{cases} -\frac{Et}{T}, & \text{if } |\tau| < \frac{T}{2}; \\ 0, & \text{otherwise.} \end{cases} \quad (40)$$

so that

$$\mathbf{e} = \begin{cases} \frac{E}{T}\hat{\mathbf{z}}, & \text{if } |\tau| < \frac{T}{2}; \\ 0, & \text{otherwise.} \end{cases} \quad (41)$$

$$\int d\tau \mathbf{e} = E\hat{\mathbf{z}} \quad (42)$$

and the ϵ field vanishes within the range $|\tau| < \frac{T}{2}$. Now, since $f^{4\mu}$ is zero (within the range of interest), we have from eqn. (12) that $ds/d\tau = m/M = \text{constant}$. The equations of motion can be reduced from

$$M \frac{d}{d\tau} \begin{pmatrix} dt/d\tau \\ dz/d\tau \end{pmatrix} = e_0 \begin{pmatrix} 0 & E/T \\ E/T & 0 \end{pmatrix} \begin{pmatrix} dt/d\tau \\ dz/d\tau \end{pmatrix}, \quad (43)$$

which is

$$M \frac{ds}{d\tau} \frac{d}{ds} \frac{ds}{d\tau} \begin{pmatrix} dt/ds \\ dz/ds \end{pmatrix} = e_0 \begin{pmatrix} 0 & E/T \\ E/T & 0 \end{pmatrix} \frac{ds}{d\tau} \begin{pmatrix} dt/ds \\ dz/ds \end{pmatrix},$$

to

$$m \frac{d}{ds} \begin{pmatrix} dt/ds \\ dz/ds \end{pmatrix} = e_0/T \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \begin{pmatrix} dt/ds \\ dz/ds \end{pmatrix}. \quad (44)$$

We therefore find that within the range $|\tau| < \frac{T}{2}$, the equations of motion are the same as for the Maxwell case, with the replacement $e \rightarrow e_0/T$, which we observe has the correct dimensionless form.

We now solve the full equations of motion. With the form of the a^3 field as given above, the ϵ field is given as

$$\epsilon^3 = f^{43} = \partial^4 a^3 = -\partial_\tau \left[-\frac{Et}{T} (\theta(\tau + \frac{T}{2}) - \theta(\tau - \frac{T}{2})) \right] \quad (45)$$

so that

$$\epsilon = \frac{Et}{T} [\delta(\tau + \frac{T}{2}) - \delta(\tau - \frac{T}{2})] \hat{z}. \quad (46)$$

We take as initial conditions $\dot{z}(-\infty) = 0$ and $\dot{t}(-\infty) = m_0/M$. The complete equations of motion now become

$$M \frac{d}{d\tau} \begin{pmatrix} dt/d\tau \\ dz/d\tau \end{pmatrix} = e_0 \begin{pmatrix} 0 & E/T \\ E/T & 0 \end{pmatrix} \begin{pmatrix} dt/d\tau \\ dz/d\tau \end{pmatrix} [\theta(\tau + \frac{T}{2}) - \theta(\tau - \frac{T}{2})] \\ + \frac{e_0 Et(\tau)}{T} \begin{pmatrix} 0 \\ \delta(\tau + \frac{T}{2}) - \delta(\tau - \frac{T}{2}) \end{pmatrix}. \quad (47)$$

The event will evolve freely for $\tau < -T/2$, and then encounter ϵ which gives \dot{z} a jump to $e_0 Et(-T/2)/MT = -m_0 e_0 E/2M^2$ and leaves \dot{t} unchanged. Since \dot{x}^2 has now changed, the event transfers mass to the ϵ field, and its new mass is

$$m(-\frac{T}{2}) = \sqrt{-M^2(\dot{x}^2)} = m_0 \sqrt{1 - (\frac{e_0 E}{2M})^2}, \quad (48)$$

for $e_0 E < 2M$. Note that if $e_0 E > 2M$, then the motion of the event becomes *spacelike* (tachyonic), and the squared mass becomes negative. At $\tau = -\frac{T}{2}^+$, we find the event at $t = -m_0 T/2M$, $\dot{t} = m_0/M$, $\dot{z} = -m_0 e_0 E/2M^2$. From $\tau = -\frac{T}{2}^+$ to $\tau = \frac{T}{2}^-$, the event will evolve in the \mathbf{e} field, according to

$$\begin{pmatrix} \dot{t}(\tau) \\ \dot{z}(\tau) \end{pmatrix} = \begin{pmatrix} \cosh \frac{e_0 E}{MT} (\tau + \frac{T}{2}) & \sinh \frac{e_0 E}{MT} (\tau + \frac{T}{2}) \\ \sinh \frac{e_0 E}{MT} (\tau + \frac{T}{2}) & \cosh \frac{e_0 E}{MT} (\tau + \frac{T}{2}) \end{pmatrix} \begin{pmatrix} \dot{t}(-\frac{T}{2}^+) \\ \dot{z}(-\frac{T}{2}^+) \end{pmatrix}. \quad (49)$$

Then at $\tau = \frac{T}{2}^-$, the event will reach

$$t(T/2) = m_0 T \left[\frac{\sinh(e_0 E/M)}{e_0 E} - \frac{\cosh(e_0 E/M)}{2M} \right]$$

$$z(T/2) = z(-\infty) + m_0 T \left[\frac{\cosh(e_0 E/M)}{e_0 E} - \frac{\sinh(e_0 E/M)}{2M} \right] \quad (50)$$

At $\tau = T/2$, the event will again encounter the ϵ field, where \dot{t} will be unchanged, while \dot{z} will jump by

$$\dot{z}\left(\frac{T}{2}^+\right) - \dot{z}\left(\frac{T}{2}^-\right) = -e_0 E t(T/2)/MT = -m_0 e_0 E/M \left[\frac{\sinh(e_0 E/M)}{e_0 E} - \frac{\cosh(e_0 E/M)}{2M} \right], \quad (51)$$

which returns \dot{z} to zero. Then for $\tau > T/2$, the event will evolve freely, with $\dot{z} = 0$, and

$$\dot{t} = \frac{m_0}{M} \left[\cosh\left(\frac{e_0 E}{M}\right) - \left(\frac{e_0 E}{M}\right) \sinh\left(\frac{e_0 E}{2M}\right) \right] \quad (52)$$

which for large $e_0 E/2M$ becomes,

$$\dot{t} = \frac{m_0}{2M} \left[1 - \frac{e_0 E}{2M} \right] e^{\left(\frac{e_0 E}{M}\right)} \quad (53)$$

and we see that the condition for pair creation is $e_0 E > 2M$, which is the condition that the evolution be spacelike when $|\tau| < \frac{T}{2}$.

The example we have given above illustrates the striking differences that may exist between the effects of the pre-Maxwell Lorentz forces and what would be expected from the Lorentz forces associated with the corresponding (through integration over τ) Maxwell field. If, on the other hand, one considers an example with adiabatic dependence of the pre-Maxwell field, the term f^{34} can be made very small. In this case, the pre-Maxwell Lorentz forces result in a motion close to what would be predicted from the Maxwell Lorentz forces (as discussed in connection with Eq. (44)). This type of example can be constructed by taking for a^3 a Gaussian function of $\frac{\tau^2}{T^2}$, with T taken sufficiently large.

To study the plane wave solutions for the source free case, we take the Fourier transform of equations (23), resulting in

$$\mathbf{k} \cdot \mathbf{e} = \sigma \kappa \epsilon^0 \quad \mathbf{k} \cdot \mathbf{h} = 0$$

$$\mathbf{k} \times \mathbf{e} - k^0 \mathbf{h} = 0 \quad \mathbf{k} \times \mathbf{h} + k^0 \mathbf{e} = \sigma \kappa \epsilon \quad (54)$$

and

$$\mathbf{k} \times \epsilon - \kappa \mathbf{h} = 0 \quad \mathbf{k} \cdot \epsilon - k^0 \epsilon^0 = 0$$

$$-\kappa \mathbf{e} + k^0 \epsilon = \mathbf{k} \epsilon^0, \quad (55)$$

where we have used the fact that $\sigma^2 = 1$ and defined the transform by

$$f(x, \tau) = \int d^4 k d\kappa e^{i(k \cdot x + \sigma \kappa \tau)} f(k, \kappa). \quad (56)$$

We remark that integration of (56) over τ (or, through the Riemann- Lebesgue lemma, in the limit $\tau \rightarrow \infty$; the static limit referred to above corresponds to taking $\kappa = 0$) selects the $\kappa = 0$ component; in this case, eqs.(54) reduce to the usual Maxwell form (for which \mathbf{e} , as well as \mathbf{h} , is perpendicular to \mathbf{k} ; they are orthogonal to each other and $\mathbf{e}^2 = \mathbf{h}^2$). Eqs. (54) and (55) decouple and it follows that ϵ becomes *parallel* to \mathbf{k} , $k^0 = |\mathbf{k}|$ and $\epsilon^0 = |\epsilon|$. In fact, in this case, the two null vectors are parallel, i.e., $\epsilon^\mu = (\epsilon^0/k^0)k^\mu$. With this “natural” limit in mind, we decompose

$$\mathbf{e} = \mathbf{e}_\perp + \mathbf{e}_\parallel \quad \epsilon = \epsilon_\perp + \epsilon_\parallel \quad (57)$$

and, in terms of the field components \mathbf{e}_\perp and ϵ_\parallel , which we take to be independent, one finds from eqs. (54) and (55) that

$$\begin{aligned} \mathbf{e} &= \mathbf{e}_\perp + \sigma\left(\frac{\kappa}{k^0}\right)\epsilon_\parallel & \mathbf{h} &= \frac{1}{k^0}\mathbf{k} \times \mathbf{e}_\perp \\ \epsilon &= \epsilon_\parallel + \left(\frac{\kappa}{k^0}\right)\mathbf{e}_\perp & \epsilon^0 &= \frac{1}{k^0}\mathbf{k} \cdot \epsilon_\parallel \\ \mathbf{k}^2 - (k^0)^2 + \sigma\kappa^2 &= 0. \end{aligned} \quad (58)$$

As in the Maxwell case, the premagnetic field \mathbf{h} is normal to the propagation vector \mathbf{k} ; the two pre-electric fields \mathbf{e} , ϵ , are in a plane normal to \mathbf{h} , but need not be normal to \mathbf{k} . In the Maxwell limit, as discussed above, $\kappa \rightarrow 0$, $\mathbf{e} \rightarrow \mathbf{e}_\perp$, $\mathbf{k}^2 - (k^0)^2 \rightarrow 0$.

The energy-momentum tensor $T^{\alpha\beta}$, which is conserved in the absence of sources, is given [3] by,

$$T^{\alpha\beta} = \lambda\left(-\frac{1}{4}g^{\alpha\beta}f^{\gamma\delta}f_{\gamma\delta} + f^{\alpha\gamma}f_\gamma^\beta\right). \quad (59)$$

In terms of the non-covariant fields, we evaluate the following components of $T^{\alpha\beta}$ for the plane wave solutions:

$$\begin{aligned} T^{00} &= \frac{\lambda}{2}[\mathbf{e}^2 + \mathbf{h}^2 + \sigma(\epsilon^2 + (\epsilon^0)^2)] \\ T^{0i} &= \lambda[\mathbf{e} \times \mathbf{h} + \sigma\epsilon^0\epsilon]^i \\ T^{44} &= \frac{\lambda}{2}[\sigma(\mathbf{e}^2 - \mathbf{h}^2) + \epsilon^2 - (\epsilon^0)^2] \\ T^{4i} &= \lambda[\epsilon^0\mathbf{e} + \epsilon \times \mathbf{h}]^i \\ T^{40} &= \lambda(\mathbf{e} \cdot \epsilon). \end{aligned} \quad (60)$$

We recall that T^{00} and $P^i = T^{0i}$ are the usual energy density and Poynting vector (three-momentum) of the fields. Moreover, the $O(3,1)$ invariant T^{44} represents the mass density of the fields, while the Poynting four-vector $S^\mu = T^{4\mu}$ is associated with the motion of mass density into time and space directions. When we evaluate these components for the plane wave solutions, we obtain the following expressions:

$$T^{00} = \lambda[\mathbf{e}_\perp^2 + \sigma\epsilon_\parallel^2]$$

$$\begin{aligned}
\mathbf{P} &= T^{00} \mathbf{k}/k^0 \\
T^{44} &= T^{00} (\kappa/k^0)^2 \\
\mathbf{S} &= T^{00} (\mathbf{k}/k^0)(\kappa/k^0) \\
S^0 &= T^{00} \kappa/k^0
\end{aligned} \tag{61}$$

These expressions demonstrate that the two Poynting vectors, \mathbf{P} and \mathbf{S} are parallel to the propagation vector \mathbf{k} . Moreover,

$$\begin{aligned}
(T^{00}, \mathbf{P}) &= (T^{00}/k^0)(k^0, \mathbf{k}) \\
(S^0, \mathbf{S}) &= (T^{00}, \mathbf{P})(\kappa/k^0)
\end{aligned} \tag{62}$$

Therefore, both $T^{0\mu}$ and $T^{4\mu}$ are in the direction of the four-momentum k^μ . Note that the O(3,1) invariant $\epsilon_\mu \epsilon^\mu$ has the form $\epsilon_\mu \epsilon^\mu = T^{44}/\lambda$. The factor κ/k^0 , which appears in (58), (61), and (62), is essentially the phase velocity in the time direction of the plane wave, that is $\Delta t/\Delta \tau$. The relationships (61) among the components of $T^{\alpha\beta}$ are also of a kinematical nature, and may be obtained from the conservation of energy-momentum, the proportionality of $T^{\alpha\mu}$ to k^μ , and the five dimensional mass-shell condition $g_{\alpha\beta} T^{4\alpha} T^{4\beta} = 0$.

In [3], it is shown that in the presence of sources, the energy-momentum tensor satisfies

$$-\partial_\beta T^{\alpha\beta}(x, \tau) = e_0 f^{\alpha\beta} j_\beta(x, \tau) \tag{63}$$

If we take for j_β , the single-particle current defined in (15),

$$j_\beta(x, \tau) = \dot{r}_\beta(\tau) \delta^4(x - r(\tau)) \tag{64}$$

then we may integrate (63),

$$\int d^4x [\partial_\beta T^{\alpha\beta}(x, \tau) + e_0 f^{\alpha\beta} j_\beta(x, \tau)] = 0$$

and, with appropriate boundary conditions for $T^{\alpha\beta}$, we obtain,

$$\left[\frac{d}{d\tau} \int d^4x T^{4\alpha}(x, \tau) \right] + e_0 f^{\alpha\beta}(r, \tau) \dot{r}_\beta(\tau) = 0. \tag{65}$$

We recognize the second term in (65) as the covariant form of the Lorentz force, and we may use (13) to rewrite (65) as

$$\frac{d}{d\tau} \left\{ \int d^4x T^{4\alpha}(x, \tau) + M[p^\alpha(\tau) - e_0 a^\alpha(r(\tau), \tau)] \right\} = 0, \tag{66}$$

which expresses the conservation of the total energy-momentum-mass of the particle and fields.

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