Particles and Events in Classical Off-Shell Electrodynamics

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Abstract

Despite the many successes of the relativistic quantum theory developed by Horwitz, et. al., certain difficulties persist in the associated covariant classical mechanics. In this paper, we explore these difficulties through an examination of the classical Coulomb problem, in the framework of off-shell electrodynamics. As the local gauge theory of a covariant quantum mechanics with evolution parameter τ, off-shell electrodynamics constitutes a dynamical theory of spacetime events, interacting through five τ-dependent pre-Maxwell potentials. We present a straightforward solution of the classical equations of motion, for a test event traversing the field induced by a "fixed" event (an event moving uniformly along the time axis at a fixed point in space). This solution is seen to be unsatisfactory, and reveals the essential difficulties in the formalism at the classical level. We then offer a new model of the particle, as a certain distribution of events on the worldline, which eliminates these difficulties and permits comparison of classical off-shell electrodynamics with the standard Maxwell theory. In this model, the "fixed" event induces a Yukawa-type potential, permitting a semi-classical identification of the pre-Maxwell time scale λ with the inverse mass of the intervening photon. Numerical solutions to the equations of motion are compared with the standard Maxwell solutions — they are seen to coincide when $\lambda \gtrsim 10^{-6}$ seconds, providing an initial estimate of this parameter. It is also demonstrated that the proposed model provides a natural interpretation for the photon mass cut-off required for the renormalizability of the off-shell quantum electrodynamics.

1 Introduction

In 1973, Horwitz and Piron [1] constructed a canonical formalism for the relativistic classical and quantum mechanics of many bodies. To formulate a generalized Hamilton's principle, they introduced a Poincaré invariant evolution parameter τ, corresponding to the ordering relation among successive events in spacetime. This covariant mechanics differs from the 'proper time method' [2 – 10] in two significant ways: first, by introducing invariant two-body potentials defined on an unconstrained eight-dimensional phase space, Horwitz and Piron relaxed the identification of the parameter τ with the proper time of the classical motion, so that particle mass becomes a dynamical quantity [11]. Second, Horwitz and Piron regard τ as a true physical time, with the status of the Newtonian time in non-relativistic mechanics. Within this framework, manifestly covariant solutions have been given for problems in scattering [12], the bound state [13] with radiative transitions [14] and Zeeman spectra [15], and statistical mechanics [16].

Turning to the question of gauge invariance, Sa'ad, Horwitz, and Arshansky [17] constructed a local gauge theory in which the gauge function depends on τ as well as the spacetime coordinates. This requirement leads to a theory of five gauge compensation fields, with explicit τ-dependence, corresponding to electromagnetic modes with continuous mass spectrum. In the resulting off-shell electrodynamics, moreover, particles and gauge fields may exchange mass, even at the classical level.

A free particle in the quantum mechanics of Horwitz and Piron satisfies the Stueckelberg equation [1, 3]

$$i\hbar\partial_\tau \psi(x,\tau) = \frac{1}{2M} p^\mu p_\mu \psi(x,\tau) \ . \tag{1}$$

The equation

$$(i\hbar\partial_\tau + \frac{e_0}{c}\phi)\ \psi(x,\tau) = \frac{1}{2M}(p^\mu - \frac{e_0}{c}a^\mu)(p_\mu - \frac{e_0}{c}a_\mu)\ \psi(x,\tau) \tag{2}$$

is invariant under local gauge transformations of the form

$$\psi(x,\tau) \longrightarrow \left[\exp \frac{ie_0}{\hbar c}\Lambda(x,\tau)\right]\ \psi(x,\tau) \tag{3}$$

when the compensation fields transform as

$$a_\mu(x,\tau) \to a_\mu(x,\tau) + \partial_\mu\Lambda(x,\tau) \qquad \phi(x,\tau) \to \phi(x,\tau) + \partial_\tau\Lambda(x,\tau) \ . \tag{4}$$

Equation (2) leads to the five dimensional conserved current

$$\partial_\mu j^\mu + \partial_\tau \rho = 0 \tag{5}$$

where

$$\rho = \left|\psi(x,\tau)\right|^2 \qquad j^\mu = \frac{-i\hbar}{2M}\left\{\psi^*(\partial^\mu - i\frac{e_0}{c}a^\mu)\psi - \psi(\partial^\mu + i\frac{e_0}{c}a^\mu)\psi^*\right\} , \tag{6}$$

so that, $\left|\psi(x,\tau)\right|^2$ may be interpreted as the probability density at τ of finding the event at x. With the summation convention

$$\lambda,\mu,\nu = 0,1,2,3 \qquad \text{and} \qquad \alpha,\beta,\gamma = 0,1,2,3,5 \tag{7}$$

and the designations

$$x^5 = c\tau \qquad \partial_5 = \frac{1}{c}\partial_\tau \qquad j^5 = c\rho \tag{8}$$

the current conservation law may be written as $\partial_\alpha j^\alpha = 0$. Since $\partial_\mu j^\mu = -\partial_\tau\rho \neq 0$, we may not identify j^μ as the source current in Maxwell's equations. However, under the boundary conditions $j^5 \to 0$, pointwise, as $\tau \to \pm\infty$, integration of (5) over τ, leads to $\partial_\mu J^\mu = 0$, where

$$J^\mu(x) = \int_{-\infty}^{\infty} d\tau \; j^\mu(x,\tau) \; . \tag{9}$$

This integration has been called concatenation [18] and links the event current j^μ with the particle current J^μ defined on the entire worldline. The quantum mechanical potential theory with $a_\mu = 0$ and $-\frac{e_0}{c}\phi = V(\sqrt{x^\mu x_\mu})$ has been solved for the standard bound state [13] and scattering [12] problems.

The classical mechanics associated with this theory is obtained by transforming the Hamiltonian found from (2) to a classical Lagrangian [19], and including the gauge invariant kinetic term for the fields proposed by Sa'ad, et. al. [17]:

$$L = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e_0}{c} \dot{x}^\mu a_\mu + e_0 \phi - \frac{\lambda}{4c} f^{\alpha\beta} f_{\alpha\beta} = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e_0}{c} \dot{x}^\alpha a_\alpha - \frac{\lambda}{4c} f^{\alpha\beta} f_{\alpha\beta} \; . \tag{10}$$

In (10), we have used $\dot{x}^5 = c$ and introduced $a_5 = \frac{1}{c}\,\phi$. The gauge invariant quantity $f_{\alpha\beta}$ is defined by

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \; . \tag{11}$$

The classical Lorentz force [19] found by variation of (10) with respect to x^μ, is given by

$$M\,\ddot{x}^\mu = \frac{e_0}{c}\; f^\mu{}_\alpha(x,\tau)\,\dot{x}^\alpha \qquad\qquad \frac{d}{d\tau}(-\frac{1}{2}M\dot{x}^2) = e_0\; f_{5\alpha}\dot{x}^\alpha \; . \tag{12}$$

Exchange of mass between particles and fields may be seen in the second of equation (12); the total mass-energy-momentum of the events and fields is, however, conserved [19]. Since particle mass is not separately conserved, pair annihilation is classically permitted.

In formally raising the index $\beta = 5$ in $f^{\mu 5} = \partial^\mu a^5 - \partial^5 a^\mu$, Sa'ad et. al. argue that the action suggests a higher symmetry containing O(3,1) as a subgroup, that is, either O(4,1) or O(3,2). They wrote the metric for the field as

$$g^{\alpha\beta} = \mathrm{diag}(-1,1,1,1,\sigma) \; , \tag{13}$$

where $\sigma = \pm 1$, depending on the higher symmetry. Variation of (10) with respect to a_α yields

$$\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda c} j^\alpha = \frac{e}{c}\; j^\alpha \qquad\qquad \epsilon^{\alpha\beta\gamma\delta\epsilon}\partial_\alpha f_{\beta\gamma} = 0 \tag{14}$$

where e_0/λ is identified as the dimensionless charge e, and the current j^α associated with an event $X^\alpha = \left(X^\mu(\tau), c\tau\right)$ is given by

$$j^\alpha(x,\tau) = c\;\frac{dX^\alpha}{d\tau}\delta^4\Big(x^\mu - X^\mu(\tau)\Big) \; . \tag{15}$$

At the quantum level, the current is given by (6).

In analogy to the concatenation of the current in (9), we see that under the boundary conditions $f^{5\mu} \to 0$, pointwise as $\tau \to \pm\infty$, we recover Maxwell's equations as

$$\partial_\nu F^{\mu\nu} = \frac{e}{c} J^\mu \qquad\qquad \epsilon^{\mu\nu\rho\lambda}\partial_\mu F_{\nu\rho} = 0 \tag{16}$$

where

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau f^{\mu\nu}(x,\tau) \qquad \text{and} \qquad A^\mu(x) = \int_{-\infty}^{\infty} d\tau \; a^\mu(x,\tau) \; . \tag{17}$$

Therefore, $a^\alpha(x,\tau)$ has been called the pre-Maxwell field. Since $e_0 a_\mu$ and eA_μ must have the same dimensions, it follows from (17) that λ (and hence $e_0 = \lambda e$) must have dimensions of time. Although the parameter λ does not appear in the field equations (14), it does appear in the Lorentz force (12) through e_0. The presense of this dimensional parameter in the equations of motion is a characteristic problem in the classical formalism.

The physical Lorentz covariance of the current j^α breaks the higher symmetry of the free field equations to O(3,1). Nevertheless, the wave equation

$$\partial_\alpha \partial^\alpha \; a^\beta = (\partial_\mu \partial^\mu + \partial_\tau \partial^\tau) \; a^\beta = (\partial_\mu \partial^\mu + \sigma \; \partial_\tau^2) \; a^\beta = -\frac{e}{c} \; j^\beta \; , \tag{18}$$

reflects the causal properties of the higher symmetry through the operator on the left hand side. The classical Green's function for (18), defined through

$$\partial_\alpha \partial^\alpha G(x,x^5) = -c\delta^4(x,x^5) \; , \tag{19}$$

is given by [20]

$$G(x,x^5) = -\frac{c}{4\pi}\delta(x^2)\delta(x^5) - \frac{c}{2\pi^2}\frac{\partial}{\partial x^2} \frac{\theta(-\sigma g_{\alpha\beta}x^\alpha x^\beta)}{\sqrt{-\sigma g_{\alpha\beta}x^\alpha x^\beta}} \; . \tag{20}$$

It follows from (18) and (19) that the potential induced by a known current is given by

$$\begin{aligned} a^\beta(x,\tau) &= -\frac{e}{c}\int d^4x' dx^{5\prime} \; \frac{1}{c} \; G(x-x', x^5 - x^{5\prime}) \; j^\beta(x', c\tau') \\ &= -\frac{e}{c}\int d^4x' d\tau \; G(x-x', c\tau - c\tau') \; j^\beta(x', c\tau') \; . \end{aligned} \tag{21}$$

Under concatenation, the first term of (20) becomes the Maxwell Green's function

$$D(x) = -\frac{1}{4\pi} \; \delta(x^2) \; , \tag{22}$$

while the second term — which induces spacelike or timelike correlations [20], depending on the signature σ — vanishes. This concatenation guarantees that the Maxwell potential is related to the Maxwell current in the usual manner:

$$
\begin{aligned}
A^\mu(x) &= \int d\tau\ a^\mu(x,\tau) \\
&= -\frac{e}{c}\int d\tau \int d^4x'd\tau'\ G(x-x',c\tau-c\tau')\ j^\mu(x',c\tau') \\
&= -\frac{e}{c}\int d^4x'd\tau'\ \left[\int d\tau G(x-x',c\tau-c\tau')\right]\ j^\mu(x',c\tau') \\
&= -\frac{e}{c}\int d^4x' D(x-x')\ J^\mu(x')\ .
\end{aligned}
\tag{23}
$$

Therefore, we will refer to the Maxwell and the correlation terms of the Green's function and the induced potentials.

The off-shell quantum electrodynamics, associated with the action

$$
\mathrm{S} = \int d^4x d\tau \left\{ \psi^*(i\partial_\tau + e_0 a_5)\psi - \frac{1}{2M}\psi^*(-i\partial_\mu - e_0 a_\mu)(-i\partial^\mu - e_0 a^\mu)\psi - \frac{\lambda}{4} f_{\alpha\beta} f^{\alpha\beta} \right\}\ , \tag{24}
$$

(here, $\hbar = c = 1$) has been worked out [21]. Manifestly covariant quantization has been given canonically [22, 21] and in path integral form [23, 21], and the perturbation theory developed [21]. The Feynman rules have been used to calculate the scattering cross section for two identical particles; this cross section reduces to the standard Klein-Gordon expression when no mass exchange is permitted [21]. For any non-zero mass exchange, the forward and reverse poles each split into two and move away from the 0 and 180 degree directions, making the total cross section finite. When the photon mass spectrum is cut off, the off-shell quantum electrodynamics is counter-term renormalizable; without the cut-off, the mass integration in the loops cannot be controlled. We will see below that this cut-off has a natural interpretation in terms of the classical considerations presented here.

We now turn to the Coulomb problem in the framework of the classical off-shell electromagnetic theory introduced above. In Section 2, we set up the classical equations of motion for an event moving in the field of an event moving uniformly along the time axis, and present a straightforward solution. This solution is discussed and seen to be plainly unsatisfactory. In Section 3, we propose a new model for the relationship between events and particles, and solve the resulting equations of motion. These results are then compared with the standard Maxwell solutions.

It is a pleasure to dedicate this paper to Professor L. P. Horwitz, whose contributions to our understanding of relativistic dynamics have re-opened this subject for a generation of students.

2 The Coulomb Problem

We begin by studying the motion of a charged event in the field produced by a second event, moving uniformly along the time axis. As in classical Rutherford scattering, we hold the inducing event "fixed" on its time axis, neglecting the field of the scattered event, and the radiation field that it would induce.

According to equation (15), the current associated with an event moving uniformly along the time axis,

$$X^0(\tau) = ct = c\tau \qquad \vec{X}(\tau) = 0 \tag{25}$$

is given by

$$\begin{aligned} j^0(x,\tau) &= c\frac{dX^0}{d\tau}\delta(x^0 - c\tau)\delta^3(\vec{x}) = c\ \delta(t-\tau)\delta^3(\vec{x}) \\ \vec{j}(x,\tau) &= 0 \\ j^5(x,\tau) &= c\frac{d(c\tau)}{d\tau}\delta(x^0 - c\tau)\delta^3(\vec{x}) = j^0(x,\tau)\ . \end{aligned} \tag{26}$$

Therefore, from (21) we will have

$$a^5(x,\tau) = a^0(x,\tau) \qquad \vec{a}(x,\tau) = 0\ , \tag{27}$$

leaving only the following non-zero components of $f^{\alpha\beta}$

$$\begin{aligned} f^{0k} &= \partial^0 a^k - \partial^k a^0 = -\partial^k a^0 \\ f^{5k} &= \partial^5 a^k - \partial^k a^5 = -\partial^k a^0 \\ f^{05} &= \partial^0 a^5 - \partial^5 a^0 = -\frac{1}{c}\left(\partial_0 a^0 + \sigma\partial_\tau a^0\right)\ . \end{aligned} \tag{28}$$

The independent components of the Lorentz force may now be written as

$$\begin{aligned} M\ \ddot{x}^0 &= \frac{e_0}{c}\ f^{0\alpha}\dot{x}_\alpha \\ &= \lambda\frac{e}{c}\ (f^{0k}\dot{x}_k + f^{05}\dot{x}_5) \\ &= -\lambda\frac{e}{c}\ \left[(\partial_k a^0)\dot{x}^k + \sigma\frac{1}{c}\ (\partial_0 + \sigma\partial_\tau)\ a^0\right] \end{aligned} \tag{29}$$

and

$$\begin{aligned} M\ \ddot{x}^k &= \frac{e_0}{c}\ f^{k\alpha}\dot{x}_\alpha \\ &= \lambda\frac{e}{c}\ (f^{k0}\dot{x}_k + f^{k5}\dot{x}_5) \\ &= -\lambda\frac{e}{c}\ (\partial_k a^0)(\dot{x}^0 - \sigma c)\ , \end{aligned} \tag{30}$$

where we used the antisymmetry of $f^{\alpha\beta}$ and $\dot{x}_5 = \sigma c$. Since at low energies, $\dot{x}^0 \sim c$, we notice that the choice of $\sigma = 1$ in (30) will rule out identification with the standard Maxwell equations of motion. We therefore choose $\sigma = -1$ and study only the case of broken O(3,2) symmetry. With this choice,

$$
\begin{aligned}
M\ \ddot{x}^0 &= -\lambda \frac{e}{c}\left[(\partial_k a^0)\dot{x}^k - \frac{1}{c}\left(\partial_t - \partial_\tau\right) a^0\right] && (31)\\
M\ \ddot{x}^k &= -\lambda e\ (\partial_k a^0)(\dot{t}+1)\ . && (32)
\end{aligned}
$$

Since λ does not appear in equations (21) or (26), the coupling of the induced field to the test event will evidently depend on this parameter.

We proceed to calculate the potential $a^0(x,\tau)$ from the Green's function (20) and the current (26). The Maxwell part of the potential is given by

$$
\begin{aligned}
a^0(x,\tau) &= -\frac{e}{c}\int d^4x' d\tau' \Big[\frac{c}{4\pi}\ \delta\Big(\ (x-x')^2\ \Big)\ \delta(c\tau - c\tau')\Big]\ \Big[c\ \delta^3(\vec{x}')\ \delta(t'-\tau')\Big]\\
&= -\frac{e}{4\pi c}\int d^3x' d(ct')\ \delta\Big((\vec{x}-\vec{x}')^2 - (x^0 - x^{0\prime})^2\Big)\ \Big[c\ \delta^3(\vec{x}')\ \delta(t'-\tau)\Big]\\
&= -\frac{ec}{4\pi}\delta\Big(R^2 - c^2(t-\tau)^2\Big)\\
&= -\frac{e}{4\pi R}\ \frac{1}{2}\Big[\delta(t-\tau-R/c) + \delta(t-\tau+R/c)\Big]\ , \qquad (33)
\end{aligned}
$$

where $R = |\vec{x}|$. As required for the Maxwell part,

$$
A^0(x) = \int d\tau a^0(x,\tau) = -\frac{e}{4\pi R}\ , \qquad (34)
$$

the concatenated potential has the form of the Coulomb potential due to a "fixed" source. The second part of the pre-Maxwell potential is given by

$$
\begin{aligned}
a_{\text{correlation}}(x,\tau) &= \frac{e}{c}\ \frac{c}{2\pi^2}\int d^4x' d\tau' \left\{\frac{\partial}{\partial(x-x')^2}\ \frac{\theta\Big((x-x')^2 - c^2(\tau-\tau')^2\Big)}{\sqrt{(x-x')^2 - c^2(\tau-\tau')^2}}\right\} \times\\
&\qquad c\ \delta^3(\vec{x}')\ \delta(t'-\tau')\\
&= \frac{ec^2}{2\pi^2}\int d\tau'\ \frac{\partial}{\partial R^2}\ \frac{\theta\Big(R^2 - c^2(t-\tau')^2 - c^2(\tau-\tau')^2\Big)}{\sqrt{R^2 - c^2(t-\tau')^2 - c^2(\tau-\tau')^2}}\\
&= \frac{ec^2}{2\pi^2}\frac{\partial}{\partial R^2}\ \int d\tau'\ \frac{\theta\Big(R^2 - c^2(t-\tau')^2 - c^2(\tau-\tau')^2\Big)}{\sqrt{R^2 - c^2(t-\tau')^2 - c^2(\tau-\tau')^2}}\ . \qquad (35)
\end{aligned}
$$

Introducing the change of variables $u = t - \tau'$ and defining $\alpha = t - \tau$, equation (35) becomes

$$
a_{\text{correlation}}(x,\tau) = \frac{ec^2}{2\pi^2}\frac{\partial}{\partial R^2}\ \int du\ \frac{\theta\Big(R^2 - c^2u^2 - c^2(\alpha-u)^2\Big)}{\sqrt{R^2 - c^2u^2 - c^2(\alpha-u)^2}}\ . \qquad (36)
$$

The θ-function in the integral restricts the limits of integration to the region between the roots $u_\pm$ of the quadratic

$$\rho(u) = R^2 - c^2u^2 - c^2(\alpha - u)^2 = R^2 - c^2\alpha^2 + 2c^2\alpha u - 2c^2u^2 \ , \tag{37}$$

which are found to be

$$u_\pm = \frac{1}{2}\left[\alpha \pm \sqrt{\frac{2R^2}{c^2} - \alpha^2}\ \right] \ . \tag{38}$$

So for $2R^2/c^2 < \alpha^2$, there are no roots and $\theta(\rho(u))$ will vanish identically. Therefore,

$$a_{\text{correlation}}(x,\tau) = \frac{ec^2}{2\pi^2}\frac{\partial}{\partial R^2}\begin{cases} \int_{u_-}^{u_+} du \frac{1}{\sqrt{\rho(u)}} & \text{if } \sqrt{2}R/c < \alpha < \sqrt{2}R/c \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

Using

$$\int du \frac{1}{\sqrt{A + Bu + Cu^2}} = \frac{-1}{\sqrt{-C}}\ \sin^{-1}\left(\frac{2Cu + B}{\sqrt{B^2 - 4AC}}\right) \tag{40}$$

we find

$$a_{\text{correlation}}(x,\tau) = \frac{ec^2}{2\pi^2}\frac{\partial}{\partial R^2}\begin{cases} \frac{-1}{\sqrt{2}}\ \sin^{-1}\left(\frac{2(-2)\ u + 2\alpha}{\sqrt{4\alpha^2 + 4(R^2/c^2 - \alpha^2)(-2)}}\right)\Bigg|_{u_-}^{u_+} & \text{if } \sqrt{2}R/c < \alpha < \sqrt{2}R/c \\ 0 & \text{otherwise} \end{cases} \tag{41}$$

and using (38) we obtain

$$\sin^{-1}\left(\frac{2(-2)\ u_\pm + 2\alpha}{\sqrt{4\alpha^2 + 4(R^2/c^2 - \alpha^2)(-2)}}\right) = \sin^{-1}\left(\frac{-(\alpha \pm \sqrt{2R^2c^2 - \alpha^2}) + \alpha}{\sqrt{2R^2c^2 - \alpha^2}}\right) = \sin^{-1}(\mp 1) \ . \tag{42}$$

This expression is independent of R, and so $a_{\text{correlation}}$ will vanish. It may be shown that the correlation term will only contribute when the inducing charge undergoes acceleration. This contribution will be treated elsewhere.

Having obtained the potential induced by the source current, we may calculate the field strengths and write the equations of motion for a test event moving in this field. The support of the potentials is restricted, by the delta-functions they contain, to the light cone of the source event and so the test event moving in the induced field will be free except when its trajectory satisfies

$$t - \tau - R/c = 0 \ . \tag{43}$$

We write the initial motion of the test event as

$$x = u\tau + s \tag{44}$$

where

$$s = (s_t, s_x, s_y, 0) \qquad \text{and} \qquad u = (u_t, u_x, 0, 0) \tag{45}$$

places the motion in the $t - x$ plane, with impact parameter s_y. To facilitate comparison with the non-relativistic case, we take the velocity

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{d\vec{x}/d\tau}{dt/d\tau} \tag{46}$$

such that $|\vec{v}| \ll c$, and so $\frac{1}{c}u_t = dt/d\tau \sim 1$. Then the interaction will occur at τ^*, given by

$$0 = t - \tau^* - R/c = \frac{1}{c}u_t\tau^* + \frac{1}{c}s_t - \tau^* - \frac{1}{c}|\vec{u}\tau^* + \vec{s}| = \frac{1}{c}s_t - \frac{1}{c}|\vec{u}\tau^* + \vec{s}| + o(v/c) \tag{47}$$

and so

$$s_t = \sqrt{(s_x + u_x\tau^*)^2 + s_y^2}\ . \tag{48}$$

Solving (48), we find

$$\tau^* = \frac{1}{u_x}\left(-s_x + \sqrt{s_t^2 - s_y^2}\right) \tag{49}$$

so that

$$R^* = R(\tau^*) = s_t \qquad x^* = x(\tau^*) = \sqrt{s_t^2 - s_y^2} \qquad y^* = y(\tau^*) = s_y \tag{50}$$

and

$$\hat{x}^* = \hat{x}(\tau^*) = \frac{1}{R^*}\vec{x}^* = \frac{1}{s_t}\left(\sqrt{s_t^2 - s_y^2}, s_y, 0\right)\ . \tag{51}$$

In the neighborhood of τ^* we have

$$M\ddot{x}^k = -\lambda e\ \partial_k\left[-\frac{e}{8\pi R}\delta(t - \tau - R/c)\right](\dot{t} + 1) \simeq -\lambda e\ \partial_k\left[-\frac{e}{8\pi R}\delta(t - \tau - R/c)\right]2 \tag{52}$$

which may be integrated as

$$\dot{\vec{x}}(\tau^* + \epsilon) - \dot{\vec{x}}(\tau^* - \epsilon) = \frac{\lambda e^2}{4\pi M}\nabla\int_{\tau^*-\epsilon}^{\tau^*+\epsilon} d\tau\frac{1}{R}\delta(t - \tau - R/c) = -\frac{\lambda e^2}{4\pi M}\frac{1}{(R^*)^2}\hat{x}^*\ . \tag{53}$$

Similarly,

$$\dot{t}(\tau^* + \epsilon) - \dot{t}(\tau^* - \epsilon) = -\frac{\lambda e^2}{Mc}\frac{1}{8\pi(R^*)^2}\ \hat{x}^* \cdot \dot{\vec{x}}^* = 0 + o(v/c)\ . \tag{54}$$

Since $\vec{x}(\tau^* - \epsilon) = \vec{u}$, the final velocity will be

$$\vec{u}' = \dot{\vec{x}}(\tau^* + \epsilon) = \vec{u} - \frac{\lambda e^2}{4\pi M}\ \frac{1}{(s_t)^3}\ \left(\sqrt{s_t^2 - s_y^2}, s_y, 0\right)\ . \tag{55}$$

Imposing conservation of energy, $\vec{u}'^2 = \vec{u}^2$,

$$\vec{u}^2 = \vec{u}^2 + \left(\frac{\lambda e^2}{4\pi M}\ \frac{1}{(s_t)^3}\right)^2 - 2\frac{\lambda e^2}{4\pi M}\ \frac{1}{(s_t)^3}\ \left(\sqrt{s_t^2 - s_y^2}, s_y, 0\right) \cdot (u_x, 0, 0) \tag{56}$$

leads to the requirement

$$0 = \frac{\lambda e^2}{4\pi M} - 2u_x s_t \sqrt{s_t^2 - s_y^2}\ . \tag{57}$$

The scattering angle is given by

$$\cos\theta = \frac{u'_x}{|\vec{u}|} = \frac{u_x - \frac{\lambda e^2}{4\pi M}\ \frac{1}{(s_t)^3}\sqrt{s_t^2 - s_y^2}}{u_x} \tag{58}$$

and using

$$\cot\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \tag{59}$$

and (57) we obtain

$$\cot\frac{\theta}{2} = \frac{s_y}{\sqrt{s_t^2 - s_y^2}}\ . \tag{60}$$

This may be compared with the asymptotic scattering angle for non-relativistic Rutherford scattering, given by [24]

$$\cot\frac{\theta}{2} = \frac{2Es_y}{e^2/4\pi} = \frac{4\pi M u^2 s_y}{e^2}\ . \tag{61}$$

Requiring equality of (60) and (61) leads to

$$\sqrt{s_t^2 - s_y^2} = \frac{e^2}{4\pi M u^2}\ , \tag{62}$$

which fixes a particular value for s_t. Notice that under this equality, (57) becomes

$$\lambda = \frac{4\pi M u s_t}{e^2}\ \sqrt{s_t^2 - s_y^2} = \frac{4\pi M u s_t}{e^2}\ \frac{e^2}{4\pi M u^2} = \frac{s_t}{u} = \frac{R(\tau^*)}{u} \tag{63}$$

so that the scattering angle will agree with the asymptotic value obtained in the standard Maxwell theory only if λ is equal to the distance at the time of interaction divided by the incoming speed.

The solution presented above is unsatisfactory for a number of reasons, aside from the essential discrepancy between the piecewise linear solution found here and the smooth acceleration expected in the Coulomb problem. Notice first that from (49) we will find no solution for τ^* if $s_t < s_y$, meaning that a test event with these initial conditions will pass the source event without interaction. In particular, two events moving together along the time axis, at separation R but no initial relative offset in time will experience no Coulomb force. The second problem is that agreement with the standard (asymptotic) scattering angle in Coulomb scattering can only be obtained by assuming a dependence of the parameter λ on the initial conditions of the experiment.

The dependence of the scattering angle on the initial condition s_t of the time coordinate exposes the underlying difficulty in the straightforward approach. The standard Maxwell current $J^\mu(x)$ has its support along the entire world line of the particle, while the support of the pre-Maxwell current is concentrated at one point of the world line; in the classical case, this point (event) depends explicitly on the time synchronization (initial condition) with respect to τ. In the quantum regime, when states are defined with sharp asymptotic masses, this τ-synchronization is completely undetermined because of the uncertainty relation between τ and mass [25]. Therefore, results in relativistic quantum mechanics and off-shell quantum electrodynamics do not suffer this dependence. Furthermore, Arshansky, Horwitz, and Lavie [18] have argued that measurements made at a spacetime point x^μ do not take place at a definite τ, but rather concatenate all events — occurring at various values of τ — which could contribute to the event at x^μ. From this point of view, the initial τ-synchronization of events in a scattering experiment (and so s_t) cannot be precisely measured, even at the classical level, and may be associated with a fundamental uncertainty. The success of the formalism in the quantum regime suggests that the formulation of the classical equations of motion be modified to take account of this uncertainty.

3 Particles as Distributions of Events

In this section, we present a model which overcomes many of the difficulties presented in the previous chapter. We wish to incorporate in the description of classical particles an uncertainty in their initial conditions with respect to τ, and we take as our starting point the observation that the Maxwell current $J^\mu(x)$ determined by measurement devices [18] is insensitive to this uncertainty. To see this, we take a normalized distribution function $\varphi(\alpha)$,

$$\int_{-\infty}^{\infty} d\alpha \; \varphi(\alpha) = 1 \; . \tag{64}$$

and replace the event current $j^\mu(x,\tau)$, given in (15) for the sharply defined event $x^\mu(\tau)$, with the current

$$\begin{aligned} j^\beta_\varphi(x,\tau) &= \int_{-\infty}^{\infty} d\alpha \; \varphi(\alpha) \; j^\beta(x,\tau-\alpha) \\ &= c\int_{-\infty}^{\infty} d\alpha \; \varphi(\alpha) \; \dot{X}^\beta(\tau-\alpha) \; \delta^4\Big(x - X(\tau-\alpha)\Big) \; . \end{aligned} \tag{65}$$

Since the concatenation integral may be shifted by $\tau \longrightarrow \tau' = \tau - \alpha$, the Maxwell current found from (65) will be

$$J^\mu_\varphi(x) = \int d\tau d\alpha \; \varphi(\alpha) \; j^\mu(x,\tau-\alpha) = \int d\alpha \; \varphi(\alpha) \times \int d\tau' \; j^\mu(x,\tau') = J^\mu(x) \; , \tag{66}$$

identical to that for the sharply defined event.

Thus, we may model a particle as a collection of events whose τ-synchronization is given by a smooth distribution. The microscopic dynamics consists of events interacting through the pre-Maxwell fields, but the pre-Maxwell current which induces those fields will be a superposition of the individual event currents.

In the particular case discussed in the previous section, the current associated with a particle moving uniformly along the time axis is given by

$$j^0_\varphi(x,\tau) = \int_{-\infty}^{\infty} d\alpha \ \varphi(\alpha) \ j^0(x,\tau-\alpha) = c \ \delta^3(\vec{x}) \ \varphi(t-\tau) \ . \tag{67}$$

The pre-Maxwell potential induced by this current is then

$$\begin{aligned} a^0_\varphi(x,\tau) &= -\frac{e}{4\pi} \int d^4x' d\tau' \ \delta^3(\vec{x}') \ \varphi(t'-\tau') \ \delta\Big(\ (x-x')^2 \ \Big)\delta(\tau-\tau') \\ &= -\frac{e}{4\pi R} \ \frac{1}{2} \Big[\varphi(t-\tau-R/c) + \varphi(t-\tau+R/c)\Big] \ , \end{aligned} \tag{68}$$

so that

$$A^0(x) = \int_{-\infty}^{\infty} d\tau \ a^0(x,\tau) = \ -\frac{e}{4\pi R} \ \frac{1}{2} \int_{-\infty}^{\infty} d\tau \ \Big(\varphi(t-\tau-R/c) + \varphi(t-\tau+R/c)\Big) = -\frac{e}{4\pi R} \tag{69}$$

as required.

A convenient choice of distribution function is

$$\varphi(\alpha) = \frac{1}{2\lambda} e^{-|\alpha|/\lambda} \ , \tag{70}$$

in which λ represents the width of the distribution. For this distribution function, the induced potential is given by

$$a^0_\varphi(x,\tau) = -\frac{e}{4\pi R} \ \frac{1}{2\lambda} \ \frac{1}{2} \ \left[e^{-\left|t-\tau-R/c\right|/\lambda} + e^{-\left|t-\tau+R/c\right|/\lambda} \right] \ . \tag{71}$$

We re-write the equations of motion (31) and (32) as

$$M \ \ddot{x}^0 \ = \ -\frac{1}{2} \ \frac{e}{c} \ \left[(\partial_k \tilde{a}^0_\varphi)\dot{x}^k - \frac{1}{c} \ (\partial_t - \partial_\tau) \ \tilde{a}^0_\varphi \right] \tag{72}$$

$$M \ \ddot{x}^k \ = \ -e \ (\partial_k \tilde{a}^0_\varphi) \frac{\dot{t}+1}{2} \ , \tag{73}$$

where

$$\tilde{a}^0_\varphi = 2\lambda \ a^0_\varphi = -\frac{e}{4\pi R} \ \frac{1}{2} \ \left[e^{-\left|t-\tau-R/c\right|/\lambda} + e^{-\left|t-\tau+R/c\right|/\lambda} \right] \tag{74}$$

now includes the factor 2λ and so has the units of A^0. In the low energy limit ($v/c \sim 0$), with $t \sim \tau$ (and $\dot{t} \sim 1$), we obtain the standard equations of motion for a particle in a classical Yukawa potential,

$$M\ \ddot{x}^0 = 0 \qquad M\ \ddot{\vec{x}} = e\ (-\nabla \tilde{a}^0_\varphi)\ . \tag{75}$$

where

$$\tilde{a}^0_\varphi(x,\tau) = -\frac{e}{4\pi R}\ e^{-R/\lambda c}\ . \tag{76}$$

Thus, the parameter λ may be estimated by the experimental precision of low energy Coulomb scattering. Clearly as λ becomes very large, corresponding to a wide distribution of events, the potential $\tilde{a}^0_\varphi$ approaches the standard Coulomb potential.

The derivatives of the potential may be found from (74)

$$\begin{aligned}
\partial_k \tilde{a}^0_\varphi(x,\tau) &= \partial_k \left\{ -\frac{e}{4\pi R}\ \frac{1}{2}\ \left[e^{-\left|t-\tau-R/c\right|/\lambda} + e^{-\left|t-\tau+R/c\right|/\lambda} \right] \right\} \\
&= \frac{e}{4\pi R^2}\ \hat{x}^k\ \frac{1}{2}\ \left[e^{-\left|t-\tau-R/c\right|/\lambda} + e^{-\left|t-\tau+R/c\right|/\lambda} \right] \\
&\qquad - \frac{e}{4\pi R}\ \frac{1}{2}\ \partial_k \left[e^{-\left|t-\tau-R/c\right|/\lambda} + e^{-\left|t-\tau+R/c\right|/\lambda} \right]\ .
\end{aligned} \tag{77}$$

Using

$$\begin{aligned}
\partial_\alpha e^{-|\xi(x,\tau)|/\lambda} &= \frac{d}{d\xi} e^{-|\xi(x,\tau)|/\lambda}\ \partial_\alpha \xi(x,\tau) \\
&= \frac{d}{d\xi} \left[\theta(\xi)\ e^{-\xi(x,\tau)/\lambda} + \theta(-\xi)\ e^{+\xi(x,\tau)/\lambda} \right]\ \partial_\alpha \xi(x,\tau) \\
&= \Bigg\{ \delta(\xi)\ e^{-\xi(x,\tau)/\lambda} - \delta(-\xi)\ e^{+\xi(x,\tau)/\lambda} \\
&\qquad + \theta(\xi)\ e^{-\xi(x,\tau)/\lambda} \left(-\frac{1}{\lambda} \right) + \theta(-\xi)\ e^{+\xi(x,\tau)/\lambda} \left(\frac{1}{\lambda} \right) \Bigg\}\ \partial_\alpha \xi(x,\tau) \\
&= -\frac{1}{\lambda}\ \epsilon\Big(\xi(x,\tau)\Big) e^{-|\xi(x,\tau)|/\lambda}\ \partial_\alpha \xi(x,\tau)\ ,
\end{aligned} \tag{78}$$

we find

$$\begin{aligned}
\partial_k \tilde{a}^0_\varphi(x,\tau) &= \frac{e}{4\pi R^2}\ \hat{x}^k\ \frac{1}{2}\ \left[e^{-\left|t-\tau-R/c\right|/\lambda} + e^{-\left|t-\tau+R/c\right|/\lambda} \right] \\
+ \frac{e}{4\pi R}\ \hat{x}^k\ \frac{1}{2} &\left[-\frac{1}{\lambda c}\ \epsilon(t-\tau-R/c)\ e^{-\left|t-\tau-R/c\right|/\lambda} + \frac{1}{\lambda c}\ \epsilon(t-\tau+R/c)\ e^{-\left|t-\tau+R/c\right|/\lambda} \right]
\end{aligned} \tag{79}$$

$$\partial_t \tilde{a}^0_\varphi(x,\tau) = -\frac{1}{\lambda}\ \frac{1}{2}\ \frac{e}{4\pi R} \left[e^{-\left|t-\tau-R/c\right|/\lambda} \epsilon(t-\tau-R/c) + e^{-\left|t-\tau+R/c\right|/\lambda} \epsilon(t-\tau+R/c) \right] \tag{80}$$

and

$$\partial_\tau a^0_\varphi(x,\tau) = +\frac{1}{\lambda}\ \frac{1}{2}\ \frac{e}{4\pi R} \epsilon(t-\tau-R/c)\ \left[e^{-\left|t-\tau-R/c\right|/\lambda} + \epsilon(t-\tau+R/c)\ e^{-\left|t-\tau+R/c\right|/\lambda} \right]\ . \tag{81}$$

Using equations (79), (80), and (81) in the equations of motion (72) and (73), we find

$$\begin{aligned} M\ddot{x}^k &= -\frac{e^2}{4\pi R^2}\,\hat{x}^k\,\frac{\dot{t}+1}{2}\;\times \\ &\frac{1}{2}\Big\{\Big[1+\frac{R}{\lambda c}\epsilon(t-\tau-R/c)\Big]e^{-\left|t-\tau-R/c\right|/\lambda}+\Big[1-\frac{R}{\lambda c}\epsilon(t-\tau+R/c)\Big]e^{-\left|t-\tau+R/c\right|/\lambda}\Big\} \end{aligned} \tag{82}$$

and

$$\begin{aligned} M\ddot{t} &= -\frac{1}{2c^2}\,\frac{e^2}{4\pi R^2}\frac{1}{2}\Big\{\Big[\hat{x}^k\dot{x}_k(1+\frac{R}{\lambda c}\epsilon(t-\tau-R/c))-\frac{2R}{\lambda}\epsilon(t-\tau-R/c)\Big]e^{-\left|t-\tau-R/c\right|/\lambda} \\ &\quad+\Big[\hat{x}^k\dot{x}_k(1-\frac{R}{\lambda c}\epsilon(t-\tau+R/c))-\frac{2R}{\lambda}\epsilon(t-\tau+R/c)\Big]e^{-\left|t-\tau+R/c\right|/\lambda}\Big\}\ . \end{aligned} \tag{83}$$

In the low energy limit, the equations of motion simplify to

$$\begin{aligned} M\ddot{x}^k &= -\frac{e^2}{4\pi R^2}\hat{x}^k\frac{1}{2}\ \Big\{\Big[1+\frac{R}{\lambda c}\epsilon(-R/c)\Big]e^{-R/\lambda c}+\Big[1-\frac{R}{\lambda c}\epsilon(+R/c)\Big]e^{-R/\lambda c}\Big\} \\ &= -\frac{e^2}{4\pi R^2}\hat{x}^k\ e^{-R/\lambda c}\ \Big[1-\frac{R}{\lambda c}\Big] \end{aligned} \tag{84}$$

and

$$\begin{aligned} M\ddot{t} &= -\frac{e^2}{4\pi R^2}\ \frac{1}{2}\ \frac{1}{2c^2}\ \Big\{\Big[\hat{x}^k\dot{x}_k(1+\frac{R}{\lambda c}\epsilon(-R/c))-\frac{2R}{\lambda}\epsilon(-R/c)\Big]e^{-R/\lambda c} \\ &\qquad+\Big[\hat{x}^k\dot{x}_k(1-\frac{R}{\lambda}\epsilon(R/c))-\frac{2R}{\lambda}\epsilon(R/c)\Big]e^{-R/\lambda c}\Big\} \\ &= -\frac{e^2}{4\pi R^2}\ \frac{1}{2c^2}\ \hat{x}^k\dot{x}_k\ \ e^{-R/\lambda c}\ \Big[1-\frac{R}{\lambda c}\Big] \\ &= o(v/c)\ . \end{aligned} \tag{85}$$

Numerical solutions for the equations of motion (82) and (83) are shown in Figures 1, 2, and 3. It may be seen that the trajectories of the test event are indistinguishable from the Maxwell case, when $\lambda \gtrsim 10^{-6}$ seconds, corresponding to a photon mass $m_\gamma \sim 10^{-9}$ eV. If we take the accepted experimental error in the photon mass as the actual mass of the photon, then $m_\gamma \simeq 6\times 10^{-16}$ eV [26], which corresponds to $\lambda \simeq 1$ second.

4 Summary and Conclusions

In their initial formulation of covariant quantum mechanics, Horwitz and Piron [1] constructed an interacting theory of events, mediated by standard τ-independent Maxwell gauge fields. This construction was later deemed incomplete [17, 27] because of the self-consistency problem: on the one hand, particle worldlines are traced out by the evolution of trajectories $x^\mu(\tau)$ under the local influence of the Maxwell field $A^\mu(x)$; on the other hand, $A^\mu(x)$ depends upon the entire particle worldline — it is induced by the Maxwell current $J^\mu(x)$ which can only be found by concatenating the instantaneous current $j^\mu(x,\tau)$, after the worldline has been traced-out. There can be no *a priori* guarantee that the resulting trajectories $x^\mu(\tau)$ will produce the fields $A^\mu(x)$. The pre-Maxwell theory, developed to insure a well-posed theory of interactions, contains fields $a^\alpha(x,\tau)$ and a time scale λ required so that λa^μ and $A^\mu = \int d\tau \ a^\mu$ will have the same units. Despite the many successes of the relativistic quantum theory, the τ-dependence of the fields and the presence of λ in the equations of motion lead to difficulties at the classical level. The model of the particle as a distribution of events along the worldline eliminates these problems, and permits comparison with the standard Maxwell theory at low energy. When the range of the photon mass spectrum is taken small enough, then the equations of motion for the off-shell electromagnetic theory coincide with those of the standard Maxwell theory, within experimental limits. The model thus, provides an initial estimate of λ — if λ is larger than about 10^{-6}, then off-shell phenomena must be observable in scattering.

Shnerb and Horwitz [22] have given an interpretation of $\hbar/\lambda c^2$ as the width of the mass distribution of the off-shell photons, yielding λc as a coherence length for the photon-matter interaction. The model offered above may be seen as extending this argument to the classical level, as well as providing a mechanism for the narrow width of the photon mass spectrum, based on the structure of matter currents. A cut-off in the photon mass spectrum was also found to be necessary for the renormalization of off-shell quantum electrodynamics [21]. The superposition of currents given in (66) leads to a similar expression for the fields,

$$a^\mu_\varphi(x,\tau) = \int_{-\infty}^{\infty} d\alpha \ \varphi(\alpha) \ a^\mu(x,\tau-\alpha) \ . \tag{86}$$

From the Fourier expansion [21]

$$a^\mu(x,\tau) = \sum_{s=\text{polarizations}} \int \frac{d^4k}{2\kappa} \left[\varepsilon^\mu_s a(k,s) e^{i(k\cdot x+\sigma\kappa\tau)/\hbar} + \varepsilon^{\mu *}_s a^*(k,s) e^{-i(k\cdot x+\sigma\kappa\tau)/\hbar}\right] \tag{87}$$

where the five-dimensional mass shell condition is $\kappa = \sqrt{k^2}$, we see that under the convolution (86), the Fourier transform $a(k,s)$ will acquire a factor

$$a(k,s) \longrightarrow a(k,s) \int d\alpha \ e^{i\kappa\alpha/\hbar} \ \varphi(\alpha) \ . \tag{88}$$

Using (70) for $\varphi(\alpha)$, we find this factor to be

$$\int d\alpha \ e^{i\kappa\alpha/\hbar} \varphi(\alpha) = \int d\alpha \ e^{i\kappa\alpha/\hbar} \ \frac{1}{2\lambda} \ e^{-|\alpha|/\lambda} = \frac{1}{1+(\lambda\kappa/\hbar)^2} \ , \tag{89}$$

which provides an adequate cut-off for the renormalization of off-shell quantum electrodynamics.

We conclude with the observation that as $\lambda \to \infty$, the photon mass vanishes, the particle current spreads out along the entire worldline, and the low energy equations of motion become identical with the standard Maxwell case. For λ large but finite, the pre-Maxwell theory remains well-posed, and the expected off-shell phenomena may be compared with the experimental limit. Such a situation suggests that evidence for the pre-Maxwell theory may be found in small deviations from standard electrodynamics.

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