HCCSP-1003-06

Coulomb Potential from Lorentz Invariance in N Dimensions

Martin Land

Department of Computer Science
Hadassah College
P. O. Box 1114, Jerusalem 91010, Israel
email: martin@multinet.net.il

Abstract

Although Maxwell theory is O(3,1)-covariant, electrodynamics only transforms invariantly between Lorentz frames for special forms of the field, and the generator of Lorentz transformations is not generally conserved. Bérard, Grandati, Lages and Mohrbach have studied the O(3) subgroup, for which they found an extension of the rotation generator that satisfies the canonical angular momentum algebra in the presence of certain Maxwell fields, and is conserved by the classical motion. The extended generator depends on the field strength, but not the potential, and so is manifestly gauge invariant. The conditions imposed on the Maxwell field by the algebra lead to a Dirac monopole solution.

In this paper, we study the generalization of the Bérard, Grandati, Lages and Mohrbach construction to the full Lorentz group in N dimensions. The requirements can be maximally satisfied in a three dimensional subspace of the full Minkowski space; this subspace can be chosen to describe either an O(3)-invariant space sector, or an O(2,1)-invariant restriction of spacetime. The field solution reduces to the Dirac monopole found in the nonrelativistic case when the O(3)-invariant subspace is selected. When an O(2,1)-invariant subspace is chosen, the field strength can be associated with a Coulomb-like potential of the type $A^\mu(x) = n^\mu/\rho$, where $\rho = (x^\mu x_\mu)^{1/2}$, similar to that used by Horwitz and Arshansky to obtain a covariant generalization of the hydrogen-like bound state. In the presence of these fields, which are determined entirely by symmetry considerations, without reference to a source equation, the extended generator is conserved under classical relativistic system evolution.

1 Introduction

The Lorentz covariance of electrodynamics can be expressed through the validity of the canonical commutation relations

$$[x^\mu, x^\nu] = 0 \qquad [p^\mu, p^\nu] = 0 \qquad [x^\mu, p^\nu] = -i\hbar g^{\mu\nu} \tag{1}$$

$$\left[x^\mu, M^{\rho\lambda}\right] = i\hbar\left(x^\lambda g^{\mu\rho} - x^\rho g^{\mu\lambda}\right) \qquad \left[p^\mu, M^{\rho\lambda}\right] = i\hbar\left(g^{\mu\rho}p^\lambda - g^{\mu\lambda}p^\rho\right) \tag{2}$$

$$\left[M^{\mu\nu},\, M^{\lambda\rho}\right] = i\hbar\left(g^{\mu\lambda}M^{\nu\rho} - g^{\mu\rho}M^{\nu\lambda} - g^{\nu\lambda}M^{\mu\rho} + g^{\nu\rho}M^{\mu\lambda}\right) \tag{3}$$

when the generator of Lorentz transformations is the operator

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{4}$$

and the electromagnetic field is introduced through minimal coupling of a gauge potential to the momentum

$$p^\mu = m\dot{x}^\mu + eA^\mu. \tag{5}$$

Using (5) to transform to the coordinate-velocity basis, the relations (1) become

$$[x^\mu, x^\nu] = 0 \qquad m[x^\mu, \dot{x}^\nu] = -i\hbar g^{\mu\nu} \tag{6}$$

and

$$m^2\left[\dot{x}^\mu, \dot{x}^\nu\right] = \left[p^\mu - eA^\mu, p^\nu - eA^\nu\right] = -i\hbar e\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = -i\hbar e F^{\mu\nu}(x)\,, \tag{7}$$

a form that prompted Feynman to seek a derivation of Maxwell's equations assuming only the commutation relations (6) and noncommutivity of the velocities, without explicit recourse to canonical momentum, gauge potential, action, or variational principle. As reported by Dyson [1], Feynman showed that posing the commutation relations,

$$\left[x^i, x^j\right] = 0 \qquad m\left[x^i, \dot{x}^j\right] = i\hbar\delta^{ij}, \tag{8}$$

among the Euclidean position $x^i(t)$ and velocity $\dot{x}^i(t) = dx^i/dt$, for $i, j = 1, 2, 3$, restricts the admissible forces in the classical Newton's second law

$$m\ddot{x}^i = F^i(t, x, \dot{x}) \tag{9}$$

to the form

$$m\ddot{x}^i = E^i(t, x) + \epsilon^{ijk}\dot{x}_j B_k(t, x) \tag{10}$$

with fields that must satisfy

$$\nabla \cdot \mathbf{B} = 0 \qquad\qquad \nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} = 0. \tag{11}$$

The velocity-dependent part of the interaction in (10) enters through

$$\left[\dot{x}^i, \dot{x}^j\right] = -\frac{i\hbar}{m^2} F^{ij}(t,x) = -\frac{i\hbar}{m^2}\epsilon^{ijk} B_k(t,x), \tag{12}$$

posed as a "naive" assumption of noncommutivity for the velocity operators, and not intended to presuppose the existence of a canonical momentum.

Although Dyson treated Feynman's derivation as something of a curiosity, his article led to small flurry of new results. Several authors observed [2] that claiming Lorentz covariance in (11) conflicts with the Euclidean assumptions in (8), and similarly, (10) cannot be interpreted as the Lorentz force of Maxwell theory. Feynman's construction was generalized to the relativistic case [3, 4] in curved N-dimensional spacetime by writing the commutation relations

$$[x^\mu, x^\nu] = 0 \qquad m[\dot{x}^\mu, x^\nu] = i\hbar\, g^{\mu\nu}(x) \qquad m\left[\dot{x}_\mu, f(x)\right] = i\hbar\, \partial_\mu f(x) \tag{13}$$

$$m^2\left[\dot{x}^\mu, \dot{x}^\nu\right] = -i\hbar F^{\mu\nu}(\tau, x) \tag{14}$$

and force relation

$$m\ddot{x}^\mu = F^\mu(\tau, x, \dot{x}), \tag{15}$$

where $\mu, \nu = 0, 1, \cdots, N-1$, and $x^\mu(\tau)$ and its derivatives are function of the Poincaré-invariant evolution parameter τ. The resulting classical system

$$m[\ddot{x}^\mu + \Gamma^{\mu\lambda\nu}\dot{x}_\lambda \dot{x}_\nu] = K^\mu(\tau, x) + F^{\mu\nu}(\tau, x)\,\dot{x}_\nu \tag{16}$$

in which the covariant derivative contains the usual affine connection

$$\Gamma_{\mu\nu\rho} = \frac{1}{2}\left(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho}\right) \tag{17}$$

and the fields satisfy

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \qquad \partial_\mu K_\nu - \partial_\nu K_\mu + \frac{\partial}{\partial\tau} F_{\mu\nu} = 0 \tag{18}$$

is equivalent to the $(N+1)$-dimensional gauge theory associated with Stueckelberg's relativistic mechanics [5, 6]. Formally extending the indices to $(N+1)$ dimensions

$$\mu, \nu, \lambda = 0, 1, \cdots, N-1 \qquad\qquad \alpha, \beta, \gamma = 0, \cdots, N \tag{19}$$

with

$$x^N = \tau \qquad \partial_\tau = \partial_N \qquad F_{\mu N} = -F_{N\mu} = K_\mu \tag{20}$$

equations (16) and (18) become

$$m[\ddot{x}^\mu + \Gamma^{\mu\lambda\nu}\dot{x}_\lambda\dot{x}_\nu] = F^{\mu\beta}(\tau, x)\dot{x}_\beta \tag{21}$$

and

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0. \tag{22}$$

If the divergence of $F_{\alpha\beta}$ exists, the homogeneous field equation (22) can be paired with

$$\partial_\beta F^{\alpha\beta} = eJ^\alpha, \tag{23}$$

which takes the form of the inhomogeneous field equation in the Stueckelberg theory. When $J^\alpha(\tau, x)$ can be identified as the Stueckelberg current, (23) reduces to standard inhomogeneous Maxwell equation in an equilibrium limit (with respect to τ) of the (4+1)-dimensional gauge theory. However, neither the existence nor the form of the divergence of $F_{\alpha\beta}$ is implied by the Feynman commutator construction.

A more general result that appeared in response to Dyson's report was the proof [7] that assumptions (8) are sufficiently strong to establish the self-adjointness of the differential equations (9), from which it follows that the derived system is *a priori* equivalent to a unique Lagrangian mechanics [8] with canonical momenta whose relationship to the velocities leads directly to (7). Thus, results found by direct calculation with Feynman's commutators should conform to a canonical formulation of electrodynamics, in its Euclidean or relativistic form. Continuing in the spirit of Feynman's "naive" inquiry, one may construct the operators

$$L^{\mu\nu} = m\left(x^\mu\dot{x}^\nu - x^\nu\dot{x}^\mu\right) \tag{24}$$

which, in light of (5), are not generally equivalent to the canonical Lorentz generators (4). The resulting commutation relations among x^μ, $\dot{x}^\mu$, and $L^{\mu\nu}$ necessarily contain terms that depend on the field strength $F^{\mu\nu}(x)$ and break the closed Lie algebra for O($N-1$,1). Bérard, Grandati, Lages and Mohrbach [9] have studied the algebra associated with Euclidean transformations of this system. Calculating commutation relations with the free angular momentum

$$L_i = m\epsilon_{ijk}x^i\dot{x}^j \tag{25}$$

the noncommutativity of the velocities in (12) contributes field-dependent terms,

$$\begin{aligned}
[x_i, L_j] &= -i\hbar\epsilon_{ijk}x_k && (26)\\
[\dot{x}_i, L_j] &= -i\hbar\epsilon_{ijk}\dot{x}^k + \frac{i\hbar}{m}\delta_{ij}(\mathbf{x}\cdot\mathbf{B}) - \frac{i\hbar}{m}x_iB_j && (27)\\
[L_i, L_j] &= -i\hbar\epsilon_{ijk}L^k - i\hbar\epsilon_{ijk}x^k(\mathbf{x}\cdot\mathbf{B}) && (28)
\end{aligned}$$

breaking the canonical angular momentum algebra. The authors argue that extending the angular momentum operator L_i to include the O(3) invariance of the total particle-field system should recover the closed Lie algebra. Introducing the extended angular momentum $\tilde{L}_i$ as the sum of the particle angular momentum L_i and a field-dependent term Q_i,

$$\tilde{L}_i = L_i + Q_i \tag{29}$$

the extended commutation relations must satisfy

$$\begin{aligned}
\left[x_i, \tilde{L}_j\right] &= -i\hbar\epsilon_{ijk}x_k && (30)\\
\left[\dot{x}_i, \tilde{L}_j\right] &= -i\hbar\epsilon_{ijk}\dot{x}^k && (31)\\
\left[\tilde{L}_i, \tilde{L}_j\right] &= -i\hbar\epsilon_{ijk}\tilde{L}^k. && (32)
\end{aligned}$$

It was shown that equations (30) to (32) may be satisfied with the choice

$$Q_i = -x_i(\mathbf{x}\cdot\mathbf{B}), \tag{33}$$

which in turn imposes a structural condition on the field $\mathbf{B}$ given by

$$x_jB_i + x_iB_j + x_jx_k\partial_iB^k = 0. \tag{34}$$

Since (34) admits a solution of the form

$$B_i = -\frac{x_i}{x^3} \tag{35}$$

the authors argue that the method has led to a magnetic monopole. Using this solution, it is shown that the total angular momentum $\tilde{L}_i$ is conserved under the classical motion. It is interesting to note that the Coulomb-like solution (35) was found from symmetry considerations alone. The divergence equation

$$\nabla\cdot\mathbf{B} = \delta^3(\mathbf{x}) \tag{36}$$

follows *a posteriori* from a particular solution to condition (34), and is not imposed by Feynman's framework of commutation relations.

In this paper, we generalize the Bérard, Grandati, Lages and Mohrbach construction to the relativistic case in N dimensions and study the Lie algebra of the operators $L^{\mu\nu}$ defined in (24). The resulting relativistic construction generalizes equations (25) through (35) and illuminates important features of the symmetric structure not made explicit in the 3-dimensional case. In section 2 we derive commutation relations for the operators (24), involving an antisymmetric tensor field $F^{\mu\nu}$ associated with the noncommutivity of the velocities $\dot{x}^{\mu}$. In section 3, closed commutation relations for the extended generators

$$\tilde{L}^{\mu\nu} = L^{\mu\nu} + Q^{\mu\nu} \tag{37}$$

are found and a choice for the field-dependent tensor operator $Q^{\mu\nu}$ is proposed that generalizes (33). The structural conditions on the field $F^{\mu\nu}$ imposed by this choice are derived, and shown to reduce to (34) in the nonrelativistic limit. In section 4, the extended operator $\tilde{L}^{\mu\nu}$ is shown to be conserved under system evolution when the field $F^{\mu\nu}$ satisfies the structural conditions. In section 5, field solutions satisfying the structural conditions are found in the general form

$$F^{\mu\nu}(x) = \frac{1}{(N-2)!}\epsilon^{\mu\nu\lambda_0\lambda_1\cdots\lambda_{N-3}}G_{\lambda_0\lambda_1\cdots\lambda_{N-3}} = \frac{1}{(N-3)!}\frac{\epsilon^{\mu\nu\lambda_0\lambda_1\cdots\lambda_{N-3}}x_{\lambda_0}U_{\lambda_1\cdots\lambda_{N-3}}}{R(x)} \tag{38}$$

where $U^{\lambda_1\cdots\lambda_{N-3}}$ is a fixed antisymmetric tensor of rank $N-3$ and $R(x)$ is a scalar radial function. In $N=4$, the field $G_{\lambda_0\lambda_1\cdots\lambda_{N-3}}$ reduces to the Liénard-Wiechert field induced by an electric charge moving uniformly with four-velocity U^{μ}, and since Levi-Cevita duality exchanges the electric and magnetic fields in the four-dimensional electromagnetic field tensor, (38) can be interpreted as the covariant form of the Dirac monopole found in the nonrelativistic case. The interpretation of the monopole in $N>4$ was discussed in [10] and will be explored at greater length in a subsequent paper focusing on its underlying gauge theory. The structural conditions on the fields $F^{\mu\nu}$ are shown to imply that

$$x_{\lambda_1}U^{\lambda_1\cdots\lambda_{N-3}} = 0, \tag{39}$$

equivalent to the requirement that x^{μ} be orthogonal to $N-3$ mutually orthogonal vectors in N dimensions. Thus, the dynamical evolution $x^{\mu}(\tau)$ is restricted to the 3-dimensional subspace normal to U, which we denote

$$x^{U} = \left\{x \mid x_{\lambda_1}U^{\lambda_1\cdots\lambda_{N-3}} = 0\right\}, \tag{40}$$

and only the three Lorentz generators that leave x^{U} invariant can be made to satisfy closed commutation relations. Naturally, this restriction has no consequences in the nonrelativistic case. In $N = 4$, the vector $U = \hat{t}$ may be taken along the time axis, thereby recovering the O(3)-invariant solution obtained by Bérard, Grandati, Lages and Mohrbach, with the radial function

$$R(x) = r^3 = \left(\mathbf{x}^2\right)^{3/2} \tag{41}$$

defined on the three space dimensions. On the other hand, by taking $U = \hat{n}$ to be spacelike, the general solution (38) becomes

$$F^{\mu\nu}(x) = \epsilon^{\mu\nu\lambda\rho} \frac{\hat{n}_\lambda \; x_\rho}{\left(x_\perp^2\right)^{3/2}} \tag{42}$$

whose support is on the O(2,1)-invariant subspace

$$x^{\hat{n}} = \{x \mid x \cdot \hat{n} = 0\} \,. \tag{43}$$

For this solution, the three Lorentz generators (two boosts and one rotation) that leave this subspace invariant will satisfy the closed Lorentz algebra. The field strength (38) is associated with a potential of the type

$$V(x) \sim \left(x_\perp^2\right)^{-1/2} \tag{44}$$

which may be seen as a relativistic generalization of the nonrelativistic Coulomb potential. A solution to the relativistic bound state problem for the scalar hydrogen atom was found [11] in the context of the Horwitz-Piron [12] formalism, using a potential of the form (44). It was shown that a discrete Schrodinger-like spectrum emerges when the dynamics are restricted to the O(2,1)-invariant subspace

$$\text{RMS}(\hat{n}) = \{x \mid \hat{n} \;\; \text{spacelike}, \quad x_\perp = x - (x \cdot \hat{n})\hat{n} \;\; \text{spacelike}\} \,. \tag{45}$$

2 Commutation Relations

2.1 Spacetime algebra

Calculating the field-dependent commutation algebra outside the comfortable realm of 3-dimensional nonrelativistic mechanics, creates two new difficulties: the proliferation of tensor indices, and the conceptual difficulty of defining the magnetic monopole in $N > 4$ dimensions. These difficulties are conveniently overcome in the spacetime algebra formalism [13, 14]. By representing the usual tensorial objects of physics as index-free elements of various rank in a Clifford algebra, the formalism achieves a high degree of notational compactness, and provides a natural extension of the four-dimensional electromagnetic duality to higher dimensions [10]. The most important rules for algebraic manipulation can be built up from the product of two vectors, which separates naturally into a symmetric part and antisymmetric part

$$ab = \tfrac{1}{2}(ab + ba) + \tfrac{1}{2}(ab - ba) = a \cdot b + a \wedge b \tag{46}$$

where the symmetric part is identified with the scalar inner product, and the rank 2 antisymmetric part is called a bivector. Continuing in this way,

$$\begin{aligned} aA_r &= a(a_1 \wedge a_2 \wedge \cdots \wedge a_r) = a \cdot A_r + a \wedge A_r && (47)\\ a \cdot A_r &= \sum\nolimits_{k=1}^{r} (-1)^{k+1}(a \cdot a_k)\ a_1 \wedge \cdots \wedge a_{k-1} \wedge a_{k+1} \wedge \cdots \wedge a_r && (48)\\ a \wedge A_r &= a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r && (49) \end{aligned}$$

The general Clifford number is then a direct sum of multivectors of rank $0, 1, \ldots, N$

$$\begin{aligned} A &= A_0 + A_1 + A_2 + A_3 + \cdots + A_N && (50)\\ &= A_0 + A_1^i \mathbf{e}_i + \tfrac{1}{2} A_2^{ij} \mathbf{e}_i \wedge \mathbf{e}_j + \cdots + \tfrac{1}{N!} A_N^{i_0 i_2 \cdots i_{N-1}} \mathbf{e}_{i_0} \wedge \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_{N-1}} && (51) \end{aligned}$$

expanded on the basis

$$\{1, \mathbf{e}_i, \mathbf{e}_i \wedge \mathbf{e}_j, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k, \cdots, \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{N-1}\}. \tag{52}$$

The dimension of the basis multivector $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_r$ is $\binom{N}{r} = \frac{N!}{r!(N-r)!}$ and so the element

$$\mathbf{i} = \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{N-1} \tag{53}$$

has dimension 1, and by multiplication takes a multivector to its Levi-Cevita dual

$$\mathbf{i}\left[\mathbf{e}_{k_1} \wedge \cdots \wedge \mathbf{e}_{k_r}\right] = g_{k_1 k_1} \cdots g_{k_r k_r} \tfrac{1}{(N-r)!}\ \epsilon^{k_1 \cdots k_r\ k_{r+1} \cdots k_N} \left[\ \mathbf{e}_{k_{r+1}} \wedge \cdots \wedge \mathbf{e}_{k_N}\right]. \tag{54}$$

The element $\mathbf{i}$, called the unit pseudoscalar, has the following properties

$$\begin{aligned} \mathbf{i}^2 &= (-1)^{\frac{N(N-1)}{2}} \det(g_{\mu\nu}) && (55)\\ a \cdot (\mathbf{i}A_r) &= (-1)^{N-1}\, \mathbf{i}\,(a \wedge A_r) && (56)\\ a \wedge (\mathbf{i}A_r) &= (-1)^{N-1}\, \mathbf{i}\,(a \cdot A_r)\,. && (57) \end{aligned}$$

2.2 Representations and notation

Generalizing the algebra of Euclidean rotations to the algebra of Lorentz transformations, we start with commutation relations (13) in flat spacetime,

$$g^{\mu\nu} = \text{diag}\left(-1, 1, \cdots, 1\right) \qquad \mu, \nu = 0, 1, \cdots, N-1 \tag{58}$$

and represent the vector operators as

$$x = x^\mu \mathbf{e}_\mu \qquad\qquad \dot{x} = \dot{x}^\mu \mathbf{e}_\mu . \tag{59}$$

The entities x and $\dot{x}$ are composed of noncommuting operator-valued components x^μ and $\dot{x}^\mu$ that commute with the basis elements of the Clifford algebra, and noncommuting basis elements $\mathbf{e}_\mu$ that commute with the operator-valued components. The second rank antisymmetric tensor fields introduced in (14) are taken to be τ-independent, and are represented as bivectors

$$F\left(x\right) = F^{\mu\nu}\left(x\right) \mathbf{e}_\mu \otimes \mathbf{e}_\nu = \frac{1}{2} F^{\mu\nu}\left(x\right) \mathbf{e}_\mu \wedge \mathbf{e}_\nu . \tag{60}$$

Exercising care with operator ordering, manifest antisymmetry permits the index-free tensor

$$L = L^{\mu\nu} \mathbf{e}_\mu \otimes \mathbf{e}_\nu \tag{61}$$

to be represented as a vector product through

$$L = m\left(x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu\right) \mathbf{e}_\mu \otimes \mathbf{e}_\nu = m x^\mu \dot{x}^\nu \left(\mathbf{e}_\mu \otimes \mathbf{e}_\nu - \mathbf{e}_\nu \otimes \mathbf{e}_\mu\right) = m\left(x \wedge \dot{x}\right). \tag{62}$$

Multivector operators may be treated as Clifford scalars, by introducing auxiliary constants

$$D = D^\mu \mathbf{e}_\mu \qquad D^\lambda = g^{\lambda\mu} \mathbf{e}_\mu \qquad D_2 = D^{(2)} \wedge D^{(1)} = D^{(2)\mu} D^{(1)\nu} \mathbf{e}_\mu \wedge \mathbf{e}_\nu \tag{63}$$

and the commutators (13) may then be expressed as

$$\left[D \cdot x, x\right] = D_\mu \left[x^\mu, x^\nu\right] \mathbf{e}_\nu = 0 \tag{64}$$

and

$$\left[D \cdot \dot{x}, x\right] = D_\mu \left[\dot{x}^\mu, x^\nu\right] \mathbf{e}_\nu = D_\mu \left(\frac{i\hbar}{m} g^{\mu\nu}\right) \mathbf{e}_\nu = \frac{i\hbar}{m} D. \tag{65}$$

Similarly (14) becomes

$$\left[D \cdot \dot{x}, \dot{x}\right] = D_\mu \left[\dot{x}^\mu, \dot{x}^\nu\right] \mathbf{e}_\nu = -\frac{i\hbar}{m^2} D_\mu F^{\mu\nu} \mathbf{e}_\nu = -\frac{i\hbar}{m^2} D \cdot F. \tag{66}$$

It follows from (65) that

$$\left[D \cdot \dot{x}, f(x)\right] = \frac{i\hbar}{m} \left(D \cdot \partial\right) f(x), \tag{67}$$

so that the commutator in this form defines a directional derivative along the vector D.

2.3 Commutation relations

Using (64) and (65) the commutation relation between the operator L and position is found to be

$$[D\cdot x, L] = m\,[D\cdot x, x\wedge \dot{x}] = m\,[D\cdot x, x]\wedge \dot{x} + mx\wedge [D\cdot x, \dot{x}] = -i\hbar x\wedge D, \qquad (68)$$

while the velocity commutator

$$\begin{aligned}[D\cdot \dot{x}, L] &= m\,[D\cdot \dot{x}, x\wedge \dot{x}] = m\,[D\cdot \dot{x}, x]\wedge \dot{x} + mx\wedge [D\cdot \dot{x}, \dot{x}] \\ &= -i\hbar \dot{x}\wedge D - \frac{i\hbar}{m} x\wedge (D\cdot F) \end{aligned} \qquad (69)$$

contains a field-dependent term, as expected. The bivector equation (69) expresses the $(N-1)(N-2)/2$ commutation relations between the operators $L^{\mu\nu}$ and the component of velocity $\dot{x}$ in the direction of the arbitrary vector D. Equations (68) and (69) can be placed into component form by taking the arbitrary direction to be

$$D^{(\mu)} = g^{\mu\nu}\mathbf{e}_\nu \qquad (70)$$

and reading the components of (68) as

$$\left[x^\lambda,\, L^{\mu\nu}\right] = -i\hbar\left(x^\mu g^{\lambda\nu} - x^\nu g^{\lambda\mu}\right) \qquad (71)$$

and the components of (69) as

$$\left[\dot{x}^\lambda,\, L^{\mu\nu}\right] = -i\hbar\left(\dot{x}^\mu g^{\lambda\nu} - \dot{x}^\nu g^{\lambda\mu}\right) - \frac{i\hbar}{m}\left(x^\mu F^{\lambda\nu} - x^\nu F^{\lambda\mu}\right). \qquad (72)$$

To obtain the commutators among the operators $L^{\mu\nu}$, it is convenient to form the scalar

$$D_2\cdot L = mD^{(2)}\cdot\left[D^{(1)}\cdot(x\wedge\dot{x})\right] = m\left[\left(D^{(1)}\cdot x\right)\left(D^{(2)}\cdot\dot{x}\right) - \left(D^{(2)}\cdot x\right)\left(D^{(1)}\cdot\dot{x}\right)\right], \qquad (73)$$

exercising care to preserve the order of x and $\dot{x}$. Using (68) and (69) and extracting L, one is easily led to

$$\begin{aligned}[D_2\cdot L, L] &= i\hbar\left[D^{(2)}\wedge\left(D^{(1)}\cdot L\right) - D^{(1)}\wedge\left(D^{(2)}\cdot L\right)\right] \\ &\quad + i\hbar x\wedge\left[\left(D^{(2)}\cdot x\right)\left(D^{(1)}\cdot F\right) - \left(D^{(1)}\cdot x\right)\left(D^{(2)}\cdot F\right)\right]. \end{aligned} \qquad (74)$$

The bivector equation (74) expresses the $(N-1)(N-2)/2$ commutation relations between the operators $L^{\mu\nu}$ and the particular operator selected by the arbitrary vectors $D^{(1)}$ and $D^{(2)}$. In component form (74) is

$$\begin{aligned}[L_{\mu\nu},\, L_{\rho\lambda}] &= i\hbar\left(g_{\mu\lambda}L_{\nu\rho} - g_{\mu\rho}L_{\nu\lambda} - g_{\nu\lambda}L_{\mu\rho} + g_{\nu\rho}L_{\mu\lambda}\right) \\ &\quad + i\hbar\left(x_\mu x_\lambda F_{\nu\rho} - x_\mu x_\rho F_{\nu\lambda} - x_\nu x_\lambda F_{\mu\rho} + x_\nu x_\rho F_{\mu\lambda}\right) \end{aligned} \qquad (75)$$

which agrees with the usual expression for the closed Lorentz algebra when $F = 0$.

3 Closed Operator Algebra

3.1 Extended generators

We seek the extended operator

$$\tilde{L} = L + Q \tag{76}$$

that satisfies the closed algebra of Lorentz generators

$$\left[D \cdot x, \tilde{L}\right] = -i\hbar x \wedge D \tag{77}$$

$$\left[D \cdot \dot{x}, \tilde{L}\right] = -i\hbar \dot{x} \wedge D \tag{78}$$

$$\left[D_2 \cdot \tilde{L}, \tilde{L}\right] = i\hbar \left[D^{(2)} \wedge \left(D^{(1)} \cdot \tilde{L}\right) - D^{(1)} \wedge \left(D^{(2)} \cdot \tilde{L}\right)\right]. \tag{79}$$

Looking for a hint from the canonical relations, one may write (78) in component form

$$\left[\dot{x}^\mu, L^{\lambda\rho} + Q^{\lambda\rho}\right] = i\hbar \left(g^{\mu\lambda}\dot{x}^\rho - g^{\mu\rho}\dot{x}^\lambda\right) \tag{80}$$

and use (5) to express the Lorentz generator (4) as

$$M^{\mu\nu} = x^\mu \left(m\dot{x}^\nu + A^\nu\right) - x^\nu \left(m\dot{x}^\mu + A^\mu\right) = L^{\mu\nu} + \left(x^\mu A^\nu - x^\nu A^\mu\right). \tag{81}$$

Then the second canonical commutation relation (2) can be put into the form

$$\left[\dot{x}^\mu, L^{\lambda\rho} + x^\lambda A^\rho - x^\rho A^\lambda\right] + \left[\dot{x}^\lambda, x^\rho A^\mu\right] - \left[\dot{x}^\rho, x^\lambda A^\mu\right] - \frac{i\hbar}{m}\left(g^{\mu\lambda}A^\rho - g^{\mu\rho}A^\lambda\right) = i\hbar \left(g^{\mu\lambda}\dot{x}^\rho - g^{\mu\rho}\dot{x}^\lambda\right) \tag{82}$$

but comparison of (82) with (80) does not immediately suggest a suitable candidate for $Q^{\lambda\rho}$, particularly one depending only on the field strength and not the gauge potential. Therefore, the operator must be found as a solution to the requirements imposed by commutation relations (68), (69) and (74). Applying (68) to (76) it follows that

$$[D \cdot x, L + Q] = [D \cdot x, L] + [D \cdot x, Q] = -i\hbar x \wedge D + [D \cdot x, Q] \tag{83}$$

and comparison with (77) leads to

$$[D \cdot x, Q] = 0 \tag{84}$$

so that Q must be independent of $\dot{x}$ and its components must commute among themselves

$$\frac{\partial}{\partial \dot{x}^\mu} Q = 0 \Rightarrow [D_2 \cdot Q, Q] = 0. \tag{85}$$

Thus (76), (78) and (69) lead to

$$[D \cdot \dot{x}, L + Q] = -i\hbar \dot{x} \wedge D - \frac{i\hbar}{m} x \wedge (D \cdot F) + [D \cdot \dot{x}, Q] = -i\hbar \dot{x} \wedge D \tag{86}$$

and the extended commutation relation is

$$\left[D \cdot \dot{x}, \tilde{L}\right] = -i\hbar \dot{x} \wedge D - \frac{i\hbar}{m} \left[x \wedge (D \cdot F) - (D \cdot \partial) Q\right], \tag{87}$$

where we have used (13). Comparing (87) with (78) leads to the first condition on Q

$$\Delta_1 = -\frac{i\hbar}{m} \left[x \wedge (D \cdot F) - (D \cdot \partial) Q\right] = 0. \tag{88}$$

To find the second condition on Q, we apply (76) to the LHS of (79)

$$\left[D_2 \cdot \tilde{L}, \tilde{L}\right] = [D_2 \cdot L, L] + [D_2 \cdot L, Q] + [D_2 \cdot Q, L] \tag{89}$$

and to the RHS of (79)

$$\begin{aligned} \left[D_2 \cdot \tilde{L}, \tilde{L}\right] &= i\hbar \left[D^{(2)} \wedge \left(D^{(1)} \cdot L\right) - D^{(1)} \wedge \left(D^{(2)} \cdot L\right)\right] \\ &\quad + i\hbar \left[D^{(2)} \wedge \left(D^{(1)} \cdot Q\right) - D^{(1)} \wedge \left(D^{(2)} \cdot Q\right)\right]. \end{aligned} \tag{90}$$

Applying (73) to (89) provides

$$\begin{aligned} [D_2 \cdot L, Q] + [D_2 \cdot Q, L] &= i\hbar \left[D^{(2)} \wedge \left(D^{(1)} \cdot Q\right) - D^{(1)} \wedge \left(D^{(2)} \cdot Q\right)\right] \\ &\quad - i\hbar x \wedge \left[\left(D^{(2)} \cdot x\right) \left(D^{(1)} \cdot F\right) - \left(D^{(1)} \cdot x\right) \left(D^{(2)} \cdot F\right)\right] \end{aligned} \tag{91}$$

so that combining (79), (74), and (91) leads to

$$\begin{aligned} \left[D_2 \cdot \tilde{L}, \tilde{L}\right] &= i\hbar \left[D^{(2)} \wedge \left(D^{(1)} \cdot \tilde{L}\right) - D^{(1)} \wedge \left(D^{(2)} \cdot \tilde{L}\right)\right] \\ &\quad - i\hbar x \wedge \left[\left(D^{(2)} \cdot x\right) \left(D^{(1)} \cdot F\right) - \left(D^{(1)} \cdot x\right) \left(D^{(2)} \cdot F\right)\right] \\ &\quad - i\hbar \left[D^{(2)} \wedge \left(D^{(1)} \cdot Q\right) - D^{(1)} \wedge \left(D^{(2)} \cdot Q\right)\right] \end{aligned} \tag{92}$$

and we arrive at the second condition on the generator Q,

$$\begin{aligned} \Delta_2 &= -i\hbar \Big\{ x \wedge \left[\left(D^{(2)} \cdot x\right) \left(D^{(1)} \cdot F\right) - \left(D^{(1)} \cdot x\right) \left(D^{(2)} \cdot F\right)\right] \\ &\quad + \left[D^{(2)} \wedge \left(D^{(1)} \cdot Q\right) - D^{(1)} \wedge \left(D^{(2)} \cdot Q\right)\right] \Big\} = 0. \end{aligned} \tag{93}$$

3.2 Choice of generator

Since $Q = Q(x)$ cannot depend on $\dot{x}$ and since the identity $D = (D \cdot \partial) x$ permits condition (88) to be put into the form

$$(D \cdot \partial) Q = x \wedge [(D \cdot \partial) x] \cdot F, \tag{94}$$

we are led to consider a generator of the form

$$Q = x \wedge (x \cdot F) - x^2 F. \tag{95}$$

Using

$$x^2 F = x \cdot (x \wedge F) + x \wedge (x \cdot F) \tag{96}$$

equation (95) can be rewritten as

$$Q = x \wedge (x \cdot F) - [x \cdot (x \wedge F) + x \wedge (x \cdot F)] = -x \cdot (x \wedge F). \tag{97}$$

The geometrical meaning of (97) can be seen by normalizing (96) as

$$F = \frac{1}{x^2} [x \cdot (x \wedge F) + x \wedge (x \cdot F)] = \hat{x} \cdot (\hat{x} \wedge F) + \hat{x} \wedge (\hat{x} \cdot F). \tag{98}$$

Since $\hat{x} \cdot F = 0$ if the unit vector $\hat{x}$ is orthogonal to the plane spanned by F, we find that (98) represents the decomposition of F into coplanar and orthogonal components

$$F_{\parallel} = \hat{x} \wedge (\hat{x} \cdot F) \qquad F_{\perp} = \hat{x} \cdot (\hat{x} \wedge F) \tag{99}$$

Comparing (99) with (97), Q can be described as the component of the field F orthogonal to the observation point x. Notice that the nonrelativistic operator (33) can similarly be interpreted as a projection.

3.3 Conditions on the field F

Applying the first condition (88) on the generator Q to the trial form (97) leads to a corresponding condition on the field F

$$\Delta_1 = -\frac{i\hbar}{m} \{x \wedge (D \cdot F) + D \cdot (x \wedge F) + x \cdot (D \wedge F) + x \cdot [x \wedge (D \cdot \partial) F]\} = 0. \tag{100}$$

The second condition (93) on the generator Q is purely algebraic

$$\begin{aligned} \Delta_2 &= i\hbar \left(D^{(1)} \cdot x\right) \left[x \wedge \left(D^{(2)} \cdot F\right) - D^{(2)} \wedge (x \cdot F)\right] \\ &+ i\hbar \left(D^{(2)} \cdot x\right) \left[D^{(1)} \wedge (x \cdot F) - x \wedge \left(D^{(1)} \cdot F\right)\right] \\ &+ i\hbar x^2 \left[D^{(2)} \wedge \left(D^{(1)} \cdot F\right) - D^{(1)} \wedge \left(D^{(2)} \cdot F\right)\right] \\ &+ i\hbar \left(D^{(2)} \wedge x\right) \left[\left(D^{(1)} \wedge x\right) \cdot F\right] \\ &- i\hbar \left(D^{(1)} \wedge x\right) \left[\left(D^{(2)} \wedge x\right) \cdot F\right] = 0 \end{aligned} \tag{101}$$

and generally satisfied when (100) is satisfied.

4 Generator Conservation

The derivative of the classical extended generator with respect to the invariant time is

$$\begin{aligned}\frac{d}{d\tau}\tilde{L} &= \dot{L}+\dot{Q} \\ &= m\frac{d}{d\tau}\left(x\wedge\dot{x}\right)+\frac{d}{d\tau}\left[-x\cdot\left(x\wedge F\right)\right] \\ &= m\left(x\wedge\ddot{x}+\dot{x}\wedge\dot{x}\right)-\dot{x}\cdot\left(x\wedge F\right)-x\cdot\left(\dot{x}\wedge F\right)-x\cdot\left(x\wedge\dot{F}\right) \\ &= m\left(x\wedge\ddot{x}\right)-\dot{x}\cdot\left(x\wedge F\right)-x\cdot\left(\dot{x}\wedge F\right)-x\cdot\left(x\wedge\dot{F}\right).\end{aligned} \tag{102}$$

Using the equations of motion for τ-independent fields

$$F^{\mu\nu}\left(x,\tau\right)=F^{\mu\nu}\left(x\right) \qquad F^{\mu N}\left(x,\tau\right)=0 \tag{103}$$

leads to

$$m\ddot{x}^{\mu}=F^{\mu\nu}\dot{x}_{\nu}=-\dot{x}_{\nu}F^{\nu\mu} \qquad \longrightarrow \qquad \ddot{x}=-\frac{1}{m}\dot{x}\cdot F \tag{104}$$

and since F only depends on τ through $x\left(\tau\right)$,

$$\dot{F}^{\mu\nu}=\dot{x}^{\lambda}\partial_{\lambda}F^{\mu\nu} \qquad \longrightarrow \qquad \dot{F}=\left(\dot{x}\cdot\partial\right)F \tag{105}$$

the equations of motion become

$$\frac{d}{d\tau}\tilde{L}=-x\wedge\left(\dot{x}\cdot F\right)-\dot{x}\cdot\left(x\wedge F\right)-x\cdot\left(\dot{x}\wedge F\right)-x\cdot\left(x\wedge\left(\dot{x}\cdot\partial\right)F\right).$$

Applying (100) with $D=\dot{x}$,

$$-x\cdot\left[x\wedge\left(\dot{x}\cdot\partial\right)F\right]=x\wedge\left(\dot{x}\cdot F\right)+\dot{x}\cdot\left(x\wedge F\right)+x\cdot\left(\dot{x}\wedge F\right) \tag{106}$$

we find

$$\begin{aligned}\frac{d}{d\tau}\tilde{L} &= -x\wedge\left(\dot{x}\cdot F\right)-\dot{x}\cdot\left(x\wedge F\right)-x\cdot\left(\dot{x}\wedge F\right) \\ &\quad +x\wedge\left(\dot{x}\cdot F\right)+\dot{x}\cdot\left(x\wedge F\right)+x\cdot\left(\dot{x}\wedge F\right) \\ &= 0\end{aligned} \tag{107}$$

so that the extended Lorentz generator $\tilde{L}$ is conserved under the classical system evolution.

5 Field Solutions

In the previous sections it was shown that one may, in principle, define an extended generator that is conserved under evolution of the particle-field system and satisfies the closed $\mathrm{O}(N-1,1)$ commutation relations. This algebraic system depends, of course, on the existence of a field $F(x)$ satisfying conditions (100) and (101). Based on the nonrelativistic case, the field solution is expected to describe a magnetic monopole. After outlining the description of standard electric and magnetic monopole Maxwell fields in N dimensions, solutions of both type are considered. The Liénard-Wiechert field for a uniformly moving charge is considered, but it is shown that neither this form nor any simple generalization can satisfy condition (100). Nevertheless, under limited circumstances, the dual of the Liénard-Wiechert field does satisfy the condition.

5.1 Spacetime algebra representation of Maxwell theory

Maxwell's equations in N dimensions are written

$$\partial F = -J \tag{108}$$

where in the standard form, the electromagnetic field strength is the bivector given in (60) and the gradient and current vectors are $\partial = \partial^\alpha \mathbf{e}_\alpha$ and $J = J^\alpha \mathbf{e}_\alpha$. The LHS of (108) then separates into the divergence and exterior derivative

$$\partial \cdot F + \partial \wedge F = -J \tag{109}$$

so that associating terms of equal rank rank leads to

$$\partial \cdot F = -J \tag{110}$$

$$\partial \wedge F = 0. \tag{111}$$

Equation (110) expresses the inhomogeneous Maxwell equation, as can be seen from the component tensor form

$$\partial \cdot F = \partial^\alpha \left(\frac{1}{2} F^{\beta\gamma} \right) \mathbf{e}_\alpha \cdot (\mathbf{e}_\beta \wedge \mathbf{e}_\gamma) = \partial^\alpha F^{\beta\gamma} \frac{1}{2} \left(g_{\alpha\beta} \mathbf{e}_\gamma - g_{\alpha\gamma} \mathbf{e}_\beta \right) = \left(\partial_\beta F^{\beta\gamma} \right) \mathbf{e}_\gamma. \tag{112}$$

Similarly, using the total antisymmetry of $\mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\gamma$ we expand (111) as

$$\partial \wedge F = \partial^\alpha \left(\frac{1}{2} F^{\beta\gamma} \right) \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\gamma = \frac{1}{3!} \left(\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} \right) \mathbf{e}_\alpha \wedge \mathbf{e}_\beta \wedge \mathbf{e}_\gamma = 0, \tag{113}$$

expressing the homogeneous Maxwell equations.

The field equations can be generalized to a duality-invariant form by writing the fields and currents as direct sums of multivectors (labeled by rank) so that, for example, with

$$F = F_{(2)} + F_{(N-2)} \tag{114}$$

$$J = J_{(1)} + J_{(3)} + J_{(N-3)} + J_{(N-1)} \tag{115}$$

the Maxwell equations (108) separate by rank as

$$\partial \cdot F_{(2)} = -J_{(1)} \qquad \partial \wedge F_{(2)} = -J_{(3)} \tag{116}$$

$$\partial \cdot F_{(N-2)} = -J_{(N-3)} \qquad \partial \wedge F_{(N-2)} = -J_{(N-1)}. \tag{117}$$

Using (56) and (57), these expressions transform under duality as

$$\mathbf{i}\left[\partial \cdot F_{(2)}\right] = (-1)^{N-1}\,\partial \wedge \left[\mathbf{i}F_{(2)}\right] = -\mathbf{i}J_{(1)} \longrightarrow \partial \wedge F'_{(N-2)} = -J'_{(N-1)} \tag{118}$$

$$\mathbf{i}\left[\partial \wedge F_{(2)}\right] = (-1)^{N-1}\,\partial \cdot \left[\mathbf{i}F_{(2)}\right] = -\mathbf{i}J_{(3)} \longrightarrow \partial \cdot F'_{(N-2)} = -J'_{(N-3)} \tag{119}$$

$$\mathbf{i}\left[\partial \cdot F_{(N-2)}\right] = (-1)^{N-1}\,\partial \wedge \left[\mathbf{i}F_{(N-2)}\right] = -\mathbf{i}J_{(N-3)} \longrightarrow \partial \wedge F'_{(2)} = -J'_{(3)} \tag{120}$$

$$\mathbf{i}\left[\partial \wedge F_{(N-2)}\right] = (-1)^{N-1}\,\partial \cdot \left[\mathbf{i}F_{(N-2)}\right] = -\mathbf{i}J_{(N-1)} \longrightarrow \partial \cdot F'_{(2)} = -J'_{(1)} \tag{121}$$

which leaves the Maxwell equations form invariant and guarantees that for any solution to (116) there is a corresponding solution for (117), and vice versa. In the special case that $N = 4$ dimensions,

$$F_{(2)} = F_{(N-2)} \qquad J_{(1)} = J_{(N-3)} \qquad J_{(3)} = J_{(N-1)} \tag{122}$$

and the generalized system reduces to the two equations (116), which are then dual to one another. In particular, given a solution to the standard Maxwell configuration in four dimensions

$$\partial \cdot F_{(2)} = -J_{(1)} \qquad \partial \wedge F_{(2)} = 0 \tag{123}$$

there will be a corresponding solution to the configuration

$$\partial \cdot F'_{(2)} = 0 \qquad \partial \wedge F'_{(2)} = -J'_{(3)} \tag{124}$$

found from (118) and (119). On the other hand, again using (56) and (57), configuration (124) can be expressed in the equivalent form

$$\partial \cdot \left(\mathbf{i}F'_{(2)}\right) = \mathbf{i}J'_{(3)} \qquad\qquad \partial \wedge \left(\mathbf{i}F'_{(2)}\right) = 0 \tag{125}$$

where $\mathbf{i}F'_{(2)}$ is seen from

$$\left(\mathbf{i}F'_{(2)}\right)^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\left(F'_{(2)}\right)_{\lambda\rho} \tag{126}$$

to be the bivector formed by exchanging the electric and magnetic fields in $F'_{(2)}$. Therefore, in four dimensions, equations (116) constitute a Maxwell theory describing point electric and magnetic monopoles on a symmetric footing, and equations (116) and (117) are the generalization of this theory to higher dimensions. A more detailed discussion of the generalized monopole and its underlying gauge theory will appear in a subsequent paper.

5.2 Liénard-Wiechert field in 4 dimensions

An electric charge moving uniformly with constant four-velocity U induces a potential $A\left(x\right)$ through the Maxwell Green's function

$$A(x) = \frac{1}{2\pi}\int d^4x'\, J\left(x'\right)\delta\left[\left(x-x'\right)^2\right] \tag{127}$$

where

$$J\left(x'\right) = \int d\tau\; U\;\delta^4\left(x' - U\tau\right). \tag{128}$$

Expanding the Green's function using

$$\delta\left(f\left(x\right)\right) = \sum_i \frac{\delta\left(x-x_i\right)}{\left|\frac{df}{dx}\right|_{x=x_i}} \tag{129}$$

one is led to the Liénard-Wiechert potential for uniform motion

$$A(x) = \frac{U}{4\pi\left[x^2 + \left(x\cdot U\right)^2\right]^{1/2}} \tag{130}$$

from which the field strength tensor follows as

$$F^{\mu\nu}\left(x\right) = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{U^\mu x^\nu - U^\nu x^\mu}{4\pi\left[x^2 + \left(x\cdot U\right)^2\right]^{3/2}}. \tag{131}$$

Since the four-velocity is timelike, with $U^2 = -1$, and the observation point x can be resolved as

$$x = -U^2 x = -U\left(U\cdot x + U\wedge x\right) \tag{132}$$

with

$$x_{\parallel} = -U\,(U\cdot x) \tag{133}$$

$$x_{\perp} = -U\,(U\wedge x) = -U^2 x + U\,(U\cdot x) = x + U\,(U\cdot x)\,, \tag{134}$$

we recognize

$$(x_{\perp})^2 = x^2 + U^2\,(U\cdot x)^2 + 2\,(U\cdot x)^2 = x^2 + (U\cdot x)^2\,. \tag{135}$$

In the index-free notation of spacetime algebra, (131) may be rewritten as

$$F\,(x) = \partial\wedge A(x) = \partial\wedge\left[\frac{U}{4\pi\,(x_{\perp}^2)^{1/2}}\right] = \frac{U\wedge x_{\perp}}{4\pi\,(x_{\perp}^2)^{3/2}} = \frac{U\wedge x}{4\pi\,(x_{\perp}^2)^{3/2}} \tag{136}$$

where the final form in (136) follows from $U\wedge x_{\parallel} = 0$.

5.3 Electric field in N dimensions

We attempt a general solution of the form

$$F\,(x) = \frac{U\wedge x}{R\,(x)} \tag{137}$$

with an arbitrary fixed vector U and undetermined radial function $R(x)$. The first condition (100) on the field is

$$x\wedge(D\cdot F) + D\cdot(x\wedge F) + x\cdot(D\wedge F) + x\cdot[x\wedge(D\cdot\partial)\,F] = 0, \tag{138}$$

in which the derivative term requires

$$(D\cdot\partial)\,F = (D\cdot\partial)\,\frac{U\wedge x}{R\,(x)} = \frac{U\wedge D}{R\,(x)} - \frac{(U\wedge x)\,(D\cdot\partial)\,R\,(x)}{R^2\,(x)} \tag{139}$$

so that

$$\begin{aligned} x\cdot[x\wedge(D\cdot\partial)\,F] &= x\cdot\left[\left(\frac{x\wedge U\wedge D}{R\,(x)} - \frac{(x\wedge U\wedge x)\,(D\cdot\partial)\,R\,(x)}{R^2\,(x)}\right)\right] \\ &= -x\cdot\left(\frac{D\wedge U\wedge x}{R\,(x)}\right), \end{aligned} \tag{140}$$

where we used $x \wedge x = 0$. Similarly, $D \cdot (x \wedge F)$ vanishes identically for this solution, so (138) becomes

$$\begin{aligned} 0 &= x \wedge \left(D \cdot \frac{U \wedge x}{R(x)} \right) + x \cdot \left(\frac{D \wedge U \wedge x}{R(x)} \right) - x \cdot \left(\frac{D \wedge U \wedge x}{R(x)} \right) \\ &= \frac{x \wedge (D \cdot (U \wedge x))}{R(x)} \\ &= \frac{x \wedge ((D \cdot U) x - U (D \cdot x))}{R(x)} \\ &= \frac{(U \wedge x)(D \cdot x)}{R(x)} \\ &= (D \cdot x) F(x) . \end{aligned} \tag{141}$$

Notice that D is arbitrary, specifying the component of velocity being commuted with the generator. Since $x \wedge x \equiv 0$ causes the second term of (140) to vanish for algebraic reasons, the trial solution (137) fails for any form of the radial function $R(x)$. Moreover, expression (137) is seen to be an unlikely candidate for the field solution, because it leads to

$$Q = -x \cdot (x \wedge F) = -x \cdot \left(x \wedge \frac{U \wedge x}{R(x)} \right) \equiv 0. \tag{142}$$

5.4 Dual field in N dimensions

Generalizing the 3-dimensional result, it may be assumed that the field $F(x)$ is given as the dual of some field $G(x)$, so that replacing

$$F(x) = \mathbf{i} G(x) \tag{143}$$

and applying (56) and (57) to requirement (100), the first condition on the field G becomes

$$\begin{aligned} 0 &= \Delta_1 = -\frac{i\hbar}{m} \Big\{ x \wedge (D \cdot \mathbf{i}G) + D \cdot (x \wedge \mathbf{i}G) + x \cdot (D \wedge \mathbf{i}G) + x \cdot [x \wedge (D \cdot \partial) \mathbf{i}G] \Big\} \\ &= -\frac{i\hbar}{m} \mathbf{i} \Big\{ x \cdot (D \wedge G) + D \wedge (x \cdot G) + x \wedge (D \cdot G) + x \wedge [x \cdot (D \cdot \partial) G] \Big\}. \end{aligned} \tag{144}$$

Since the field F was introduced in (14) as a bivector, the Levi-Cevita dual field G must be an $(N-2)$-vector in N dimensions. Generalizing expression (137) the general solution for $G(x)$ may be written as

$$G(x) = \frac{x \wedge U}{R(x)} \tag{145}$$

where U is a fixed multivector of rank $N-3$. Now

$$(D \cdot \partial) G = (D \cdot \partial) \frac{x \wedge U}{R(x)} = \frac{D \wedge U}{R(x)} - \frac{(x \wedge U)(D \cdot \partial) R(x)}{R^2(x)} \tag{146}$$

so that

$$x \wedge [x \cdot (D \cdot \partial) G] = \frac{(x \cdot D)(x \wedge U) - x \wedge D \wedge (x \cdot U)}{R(x)} - \frac{x^2 (x \wedge U)(D \cdot \partial) R(x)}{R^2(x)}. \quad (147)$$

The other terms in (144) are

$$x \wedge (D \cdot G) = \frac{(D \cdot x)(x \wedge U)}{R(x)} \quad (148)$$

$$D \wedge (x \cdot G) = \frac{x^2 D \wedge U - D \wedge x \wedge (x \cdot U)}{R(x)} \quad (149)$$

$$x \cdot (D \wedge G) = \frac{(x \cdot D)(x \wedge U) - x^2 D \wedge U + D \wedge x \wedge (x \cdot U)}{R(x)}. \quad (150)$$

Assembling the terms, condition (100) takes the form

$$0 = \Delta_1 = -\frac{i\hbar}{m}\mathbf{i}\left\{\frac{(x \cdot D)(x \wedge U)}{R(x)} - \frac{x^2 (x \wedge U)(D \cdot \partial) R(x)}{R^2(x)} - \frac{x \wedge D \wedge (x \cdot U)}{R(x)}\right\} \quad (151)$$

in which the derivative term makes the form of $R(x)$ significant. Taking the radial function to be

$$R(x) = \left(x^2\right)^{3/2} \quad (152)$$

so that

$$\frac{(D \cdot \partial) R(x)}{R^2(x)} = 3\frac{D \cdot x}{x^2 R(x)} \quad (153)$$

leads to

$$0 = \Delta_1 = -\frac{i\hbar}{m}\,\mathbf{i}\left[\frac{D \wedge x \wedge (x \cdot U)}{R(x)}\right] = -\frac{i\hbar}{m}\,\mathbf{i}\,\frac{D \wedge U_\parallel}{\left(x^2\right)^{1/2}} \quad (154)$$

where we used (99) to write

$$x \wedge (x \cdot U) = x^2 U_\parallel. \quad (155)$$

Again, since the constant vector D is arbitrary, the condition imposed by (154) is satisfied when the dynamical evolution $x^\mu(\tau)$ is restricted to the subspace defined by

$$x(\tau) \in x^U = \{x \mid x \cdot U = 0\}, \quad (156)$$

equivalent to the requirement that x^μ be orthogonal to $N-3$ mutually orthogonal vectors in N dimensions. We conclude that the Bérard, Grandati, Lages and Mohrbach construction may be generalized to any 3-dimensional subspace of N-dimensional Minkowski space. The generators of the O(2,1) or O(3) subgroup of O($N-1$,1) that leave the subspace x^U invariant have an extension that is dynamically conserved and satisfies closed commutation relations.

The second condition on the field $F(x)$ was given in (101). With the replacement (143) this condition becomes

$$\begin{aligned}\Delta_2 \;=\;& i\hbar\left(D^{(1)}\cdot x\right)\left[\mathbf{i}x\cdot\left(D^{(2)}\wedge G\right)-\mathbf{i}D^{(2)}\cdot(x\wedge G)\right]\\ &+i\hbar\left(D^{(2)}\cdot x\right)\left[\mathbf{i}D^{(1)}\cdot(x\wedge G)-\mathbf{i}x\cdot\left(D^{(1)}\wedge G\right)\right]\\ &+i\hbar x^2\left[\mathbf{i}D^{(2)}\cdot\left(D^{(1)}\wedge G\right)-\mathbf{i}D^{(1)}\cdot\left(D^{(2)}\wedge G\right)\right]\\ &+i\hbar\left(D^{(2)}\wedge x\right)\left(\mathbf{i}D^{(1)}\wedge(x\wedge G)\right)-i\hbar\left(D^{(1)}\wedge x\right)\left(\mathbf{i}D^{(2)}\wedge(x\wedge G)\right)\end{aligned}\tag{157}$$

Applying (145) and using $x\wedge x\equiv 0$, this reduces to

$$\begin{aligned}\Delta_2 \;=\;& i\hbar\frac{\mathbf{i}}{R(x)}\left(D^{(1)}\cdot x\right)\left[x\cdot\left(D^{(2)}\wedge x\wedge U\right)\right]-i\hbar\left(D^{(2)}\cdot x\right)\left[x\cdot\left(D^{(1)}\wedge x\wedge U\right)\right]\\ &+i\hbar\frac{\mathbf{i}x^2}{R(x)}\left[D^{(2)}\cdot\left(D^{(1)}\wedge x\wedge U\right)-D^{(1)}\cdot\left(D^{(2)}\wedge x\wedge U\right)\right],\end{aligned}\tag{158}$$

so that expanding the inner products and restricting the dynamics to the subspace $x^U=\{x\mid x\cdot U=0\}$ leads to

$$\Delta_2=i\hbar\frac{\mathbf{i}x^2}{R(x)}\left[\left(D^{(1)}\wedge x\right)\wedge\left(D^{(2)}\cdot U\right)-\left(D^{(2)}\wedge x\right)\wedge\left(D^{(1)}\cdot U\right)\right].\tag{159}$$

The arbitrary vectors $D^{(1)}$ and $D^{(2)}$ in (101) specify which components of the Lorentz generator are being commuted with the index-free Lorentz generator tensor. Since (154) restricts the dynamical components to the subspace $x^U=\{x\mid x\cdot U=0\}$ it is reasonable to limit attention to the components for which $D^{(1)}\cdot U=D^{(2)}\cdot U=0$, in which case (159) is satisfied.

Although the choice of trial solution in (143) and (145) was inspired by the familiar relativistic generalization of the Coulomb interaction, the form of the radial function (152) that determines the field

$$F(x)=\mathbf{i}\frac{x\wedge U}{\left(x^2\right)^{3/2}}\tag{160}$$

was imposed, as in the nonrelativistic case, entirely by symmetry considerations, without reference to a source equation or Green's function, and identical in any number of dimensions. Nevertheless, the field is seen *a posteriori* to be of a restricted Liénard-Wiechert form, simplifying the evaluation of the divergence and exterior derivatives and enabling a classification of the solution as a monopole. Since the case of $N=4$ is unique, this case is treated separately.

5.5 Monopole field in 4 dimensions

In $N = 4$, the multivector U is just a four-vector and $G(x)$ in (145) is, up to a sign, the Liénard-Wiechert field induced by a uniformly moving electric charge

$$G(x) = \frac{x \wedge U}{(x^2)^{3/2}} \tag{161}$$

valid in the subspace for which

$$x^U = \{x \mid x \cdot U = 0\}. \tag{162}$$

Since $G(x)$ is the exterior derivative of the Liénard-Wiechert potential (130), its divergence is can be found from

$$\partial \cdot G(x) = \partial \cdot [\partial \wedge A(x)] = \partial^2 A. \tag{163}$$

By construction, $A(x)$ was found as the solution to the wave equation

$$\partial^2 A = -J(x) \tag{164}$$

with current $J(x)$ given by (128), so that

$$\partial \cdot G(x) = -J(x) = -\int d\tau\, U\, \delta^4(x - U\tau) \tag{165}$$

The exterior derivative

$$\partial \wedge G(x) = \partial \wedge [\partial \wedge A(x)] = [\partial \wedge \partial] \wedge A(x) \equiv 0 \tag{166}$$

is verified by direct calculation on (161) using $x \wedge x = \partial \wedge x = 0$. Therefore, the field $F(x)$ must satisfy

$$\partial \cdot F(x) = \partial \cdot [\mathbf{i}G(x)] = (-1)^{N-1}\, \mathbf{i}\, [\partial \wedge G(x)] = 0 \tag{167}$$

$$\partial \wedge F(x) = \partial \wedge [\mathbf{i}G(x)] = (-1)^{N-1}\, \mathbf{i}\, [\partial \cdot G(x)] = (-1)^N\, \mathbf{i}J(x) \tag{168}$$

which, in light of the discussion in section 5.1, can be interpreted as the field equations for the uniform magnetic monopole current

$$J_{(3)}(x) = \int d\tau\, \mathbf{i}U\, \delta^4(x - U\tau). \tag{169}$$

The nonrelativistic case is recovered by choosing the unit vector along the time axis $U = \mathbf{e}_0$, imposing the restriction of the dynamics to

$$x(\tau) = (0, \mathbf{x}) \tag{170}$$

so that

$$G(x) = \frac{\mathbf{e}_0 \wedge \mathbf{x}}{r^3} = \frac{x^1}{r^3}\mathbf{e}_0 \wedge \mathbf{e}_1 + \frac{x^2}{r^3}\mathbf{e}_0 \wedge \mathbf{e}_2 + \frac{x^3}{r^3}\mathbf{e}_0 \wedge \mathbf{e}_3 \tag{171}$$

and

$$F(x) = E^i \mathbf{e}_0 \wedge \mathbf{e}_i + \tfrac{1}{2}\epsilon^{ijk} B_i \mathbf{e}_j \wedge \mathbf{e}_k = \mathbf{i}\frac{\mathbf{e}_0 \wedge \mathbf{x}}{r^3} = -\frac{x^1}{r^3}\mathbf{e}_2 \wedge \mathbf{e}_3 + \frac{x^2}{r^3}\mathbf{e}_1 \wedge \mathbf{e}_3 - \frac{x^3}{r^3}\mathbf{e}_1 \wedge \mathbf{e}_2 \tag{172}$$

where

$$r = \left[\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right]^{1/2}. \tag{173}$$

The noncovariant fields can be read-off from (172) as

$$\mathbf{E} = 0 \qquad \mathbf{B} = -\frac{1}{r^3}\left(x^1, x^2, x^3\right) \tag{174}$$

reproducing the magnetic monopole of (35). Since $x \cdot U$ is a scalar in four dimensions, the extra term in the commutation relation for velocity is given in (154) as

$$\Delta_1 = \frac{i\hbar}{m}\frac{(x \cdot U)}{R(x)}\mathbf{i}(D \wedge x). \tag{175}$$

Splitting this expression into rotations and boosts as

$$\begin{aligned} \Delta_1 &= \frac{i\hbar}{m}\frac{x^0}{R(x)}\mathbf{i}\left[\left(D^0\mathbf{e}_0 + \mathbf{D}\right) \wedge \left(x^0\mathbf{e}_0 + \mathbf{x}\right)\right] &(176)\\ &= \frac{i\hbar}{m}\frac{x^0}{R(x)}\left[\left(D^0 x^i - x^0 D^i\right)\mathbf{i}(\mathbf{e}_0 \wedge \mathbf{e}_i) + \mathbf{i}(\mathbf{D} \wedge \mathbf{x})\right], &(177)\end{aligned}$$

and using

$$\mathbf{i}(\mathbf{e}_\mu \wedge \mathbf{e}_\nu) = \epsilon_{\mu\nu\lambda\rho} g^{\lambda\sigma} g^{\rho\zeta} \mathbf{e}_\sigma \wedge \mathbf{e}_\zeta, \tag{178}$$

leads to

$$\Delta_1 = \frac{i\hbar}{m}\frac{x^0}{R(x)}\left[\tfrac{1}{2}\epsilon^{0ijk}\left(D_0 x_i - D_i x_0\right)\mathbf{e}_j \wedge \mathbf{e}_k + \left(D_i x_j - D_j x_i\right)\mathbf{e}_0 \wedge \mathbf{e}_k\right]. \tag{179}$$

The components of the commutation relations for velocity with the O(3,1) generators can be read off as

$$\left[\dot{x}^i, \tilde{L}^{jk}\right] = i\hbar\left(g^{ij}\dot{x}^k - g^{ik}\dot{x}^j\right) - \frac{i\hbar}{m}\epsilon^{ijk}\frac{x^0 x_0}{R(x)} \qquad \left[\dot{x}^0, \tilde{L}^{jk}\right] = 0 \tag{180}$$

and

$$\left[\dot{x}^i, \tilde{L}^{0k}\right] = i\hbar\left(g^{i0}\dot{x}^k - g^{ik}\dot{x}^0\right) - \frac{i\hbar}{m}\epsilon^{ijk}\frac{x^0 x_j}{R(x)} \qquad \left[\dot{x}^0, \tilde{L}^{0k}\right] = i\hbar g^{00}\dot{x}^k, \tag{181}$$

which, under the restriction $x \cdot U = x^0 = 0$ of (162), become the closed relations of O(3). Similarly, applying the restriction (170) to (159), the commutation relations among the generators in component form becomes

$$\left[\tilde{L}^{\mu\nu}, \tilde{L}^{\lambda\rho}\right] = i\hbar\left\{g^{\mu\lambda}\tilde{L}^{\nu\rho} - g^{\mu\rho}\tilde{L}^{\nu\lambda} - g^{\nu\lambda}\tilde{L}^{\mu\rho} + g^{\nu\rho}\tilde{L}^{\mu\lambda}\right\} + \Delta_2^{\mu\nu\lambda\rho} \tag{182}$$

where

$$\Delta_2^{\mu\nu\lambda\rho} = 2i\hbar \frac{x^2}{R(x)} \left[g_0^\mu \epsilon^{\nu\delta\lambda\rho} x_\delta - g_0^\nu \epsilon^{\mu\delta\lambda\rho} x_\delta \right].$$

Dividing the generators, the algebra of the boosts is seen to be broken

$$\begin{aligned} \left[\tilde{L}^{0j},\, \tilde{L}^{\lambda\rho} \right] &= i\hbar \left\{ g^{0\lambda} \tilde{L}^{j\rho} - g^{0\rho} \tilde{L}^{j\lambda} - g^{j\lambda} \tilde{L}^{0\rho} + g^{j\rho} \tilde{L}^{0\lambda} \right\} + \Delta_2^{0j\lambda\rho} \\ &= i\hbar \left\{ g^{0\lambda} \tilde{L}^{j\rho} - g^{0\rho} \tilde{L}^{j\lambda} - g^{j\lambda} \tilde{L}^{0\rho} + g^{j\rho} \tilde{L}^{0\lambda} \right\} + \frac{2i\hbar x^2}{R(x)} \epsilon^{j\delta\lambda\rho} x_\delta \end{aligned} \tag{183}$$

while the rotation generators obey the closed canonical relations

$$\begin{aligned} \left[\tilde{L}^{ij},\, \tilde{L}^{\lambda\rho} \right] &= i\hbar \left\{ g^{i\lambda} \tilde{L}^{j\rho} - g^{i\rho} \tilde{L}^{j\lambda} - g^{j\lambda} \tilde{L}^{i\rho} + g^{j\rho} \tilde{L}^{i\lambda} \right\} + \Delta_2^{ij\lambda\rho} \\ &= i\hbar \left\{ g^{i\lambda} \tilde{L}^{j\rho} - g^{i\rho} \tilde{L}^{j\lambda} - g^{j\lambda} \tilde{L}^{i\rho} + g^{j\rho} \tilde{L}^{i\lambda} \right\} \end{aligned} \tag{184}$$

Since only those generators that leave the vector $U = \mathbf{e}_0$ invariant enjoy closed commutations relations, the algebra for the O(3) rotation generators closes for this choice of vector U, but does not close for the boost generators. Thus, the nonrelativistic result may be understood as equivalent to the maximal relativistic result for this choice of unit vector.

We may construct a different kind of solution by choosing $U = \mathbf{e}_3$ along the z-axis. Then, from

$$G(x) = \frac{x \wedge \mathbf{e}_3}{\rho^3} = \frac{x^0}{\rho^3} \mathbf{e}_0 \wedge \mathbf{e}_3 + \frac{x^1}{\rho^3} \mathbf{e}_1 \wedge \mathbf{e}_3 + \frac{x^2}{\rho^3} \mathbf{e}_2 \wedge \mathbf{e}_3 \tag{185}$$

the field

$$F(x) = E^i \mathbf{e}_0 \wedge \mathbf{e}_i + \tfrac{1}{2} \epsilon^{ijk} B_i \mathbf{e}_j \wedge \mathbf{e}_k = \mathbf{i} \frac{x \wedge \mathbf{e}_3}{\rho^3} = \frac{x^0}{\rho^3} \mathbf{e}_1 \wedge \mathbf{e}_2 + \frac{x^1}{\rho^3} \mathbf{e}_0 \wedge \mathbf{e}_2 - \frac{x^2}{\rho^3} \mathbf{e}_0 \wedge \mathbf{e}_1 \tag{186}$$

with

$$\rho = \left[-\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 \right]^{1/2}, \qquad \rho^2 > 0 \tag{187}$$

generalizes the spatial separation in the action-at-a-distance problems of nonrelativistic mechanics in the subspace

$$x = \left(x^0, x^1, x^2, 0\right) \in x^{\mathbf{e}_3} = \{ x \mid x \cdot \mathbf{e}_3 = 0 \} \tag{188}$$

invariant under the corresponding O(2,1) subgroup of the full Lorentz group. The field strengths are

$$\mathbf{E} = \frac{1}{\rho^3} \left(-x^2, x^1, 0\right) \qquad \mathbf{B} = \frac{1}{\rho^3} \left(0, 0, x^0\right). \tag{189}$$

In this case, subject to the restriction (188), the extra terms in the velocity relations are

$$\Delta_1 = \frac{i\hbar}{m} \frac{x^3}{R(x)} \mathbf{i} \left(D \wedge x\right) = 0 \tag{190}$$

$$\Delta_2^{\mu\nu\lambda\rho} = 2i\hbar \frac{x^2}{R(x)} \left[g_3^\mu \epsilon^{\nu\delta\lambda\rho} x_\delta - g_3^\nu \epsilon^{\mu\delta\lambda\rho} x_\delta \right], \tag{191}$$

so that closed commutation relations hold for the velocities and among the O(2,1) generators $\tilde{L}^{01}$, $\tilde{L}^{02}$, and $\tilde{L}^{12} = \tilde{L}_3$, while the algebra of the generators is broken by field dependent terms for the boost $\tilde{L}^{03}$ and the rotations $\tilde{L}^{31} = \tilde{L}_2$ and $\tilde{L}^{23} = \tilde{L}_1$.

5.6 Monopole field in higher dimensions

In dimensions higher than four, the multivector U is of rank $N-3>1$, and so the field $G(x)$ is of rank $N-2>2$. Labeling the fields and currents by rank, as in (116) and (117), the solution given by (145) satisfies

$$\begin{aligned} \partial \cdot G_{(N-2)} &= -J_{(N-3)} \qquad & \partial \wedge G_{(N-2)} = 0 \qquad (192) \\ \partial \cdot G_{(2)} &= 0 \qquad & \partial \wedge G_{(2)} = 0 \qquad (193) \end{aligned}$$

where $G_{(2)} = 0$ in this configuration. The dual system is then

$$\begin{aligned} \partial \cdot F_{(2)} &= 0 \qquad & \partial \wedge F_{(2)} = -\left(-1\right)^{N-1} \mathbf{i} J_{(N-3)} = -J'_{(3)} \qquad (194) \\ \partial \cdot F_{(N-2)} &= 0 \qquad & \partial \wedge F_{(N-2)} = 0 \qquad (195) \end{aligned}$$

where $F_{(N-2)} = 0$ follows from $G_{(2)} = 0$, and the field $F_{(2)} = \mathbf{i} G_{(N-2)}$ represents a magnetic monopole. The restriction (156) permits the extraction of the four-dimensional result from this higher dimensional system, by partitioning the N-dimensional space into the usual four-dimensional spacetime and $N-4$ 'extra dimensions' according to

$$U = n \wedge \mathbf{e}_4 \wedge \cdots \wedge \mathbf{e}_{N-1} = n \wedge \tilde{U} \qquad (196)$$

where

$$n = n^{\mu} \mathbf{e}_{\mu} \ , \ \mu = 0,1,2,3 \qquad \tilde{U} = \mathbf{e}_4 \wedge \cdots \wedge \mathbf{e}_{N-1}. \qquad (197)$$

Under this partition, (145) becomes

$$G\left(x\right) = \frac{x \wedge n \wedge \tilde{U}}{R\left(x\right)} = \left[\frac{x \wedge n}{\left(x^2\right)^{3/2}} \right] \wedge \tilde{U} \qquad (198)$$

subject to the restriction

$$x\left(\tau\right) \in x^U = \left\{ x \mid x \cdot U = x \cdot \left(n \wedge \tilde{U} \right) = \left(x \cdot n \right) \tilde{U} - n \wedge \left(x \cdot \tilde{U} \right) = 0 \right\} \qquad (199)$$

and the field $G(x)$ in (198) can be recognized as the 4-dimensional field (161) multiplied by the constant $\tilde{U}$. Choosing

$$x\left(\tau\right) = x^{\mu} \mathbf{e}_{\mu} = \left(x^0, x^1, x^2, x^3, 0, \cdots, 0 \right) \qquad (200)$$

the restriction (199) reduces to the 4-dimensional expression

$$x(\tau) \in x^U = \{x \mid (x \cdot n) = 0\} \tag{201}$$

and from (143) the general solution is

$$F(x) = \mathbf{i}\frac{x \wedge n \wedge \tilde{U}}{(x^2)^{3/2}}. \tag{202}$$

By similarly partitioning the unit pseudoscalar into the usual 4-dimensional spacetime and $N-3$ 'extra dimensions'

$$\mathbf{i} = \mathbf{e}_0 \wedge \cdots \wedge \mathbf{e}_{N-1} = (\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)(\mathbf{e}_4 \cdots \mathbf{e}_{N-1}) = \mathbf{i}_4\, \tilde{U} \tag{203}$$

the general field becomes

$$F(x) = \mathbf{i}_4\, \tilde{U}\frac{x \wedge n \wedge \tilde{U}}{(x^2)^{3/2}} = \mathbf{i}_4 \frac{x \wedge n}{(x^2)^{3/2}} \tilde{U}^2 = \left[g_{44} \cdots g_{N-1,N-1}(-1)^{\frac{(N-4)(N-5)}{2}}\right] \mathbf{i}_4 \frac{x \wedge n}{(x^2)^{3/2}} \tag{204}$$

which is, up to a sign, the solution (161). For this solution, the extended generators of O($N-1$,1) satisfy the closed commutation relations, up to extra terms of the form

$$\begin{aligned}
\Delta_2 &= i\hbar \frac{\mathbf{i}x^2}{R(x)} \left\{ \left(D^{(1)} \wedge x\right) \wedge \left[D^{(2)} \cdot \left(n \wedge \tilde{U}\right)\right] - \left(D^{(2)} \wedge x\right) \wedge \left[D^{(1)} \cdot \left(n \wedge \tilde{U}\right)\right]\right\} \\
&= i\hbar \frac{\mathbf{i}x^2}{R(x)} \left(D^{(1)} \wedge x\right) \wedge \left[\left(D^{(2)} \cdot n\right)\tilde{U} - n \wedge \left(D^{(2)} \cdot \tilde{U}\right)\right] \\
&\quad - i\hbar \frac{\mathbf{i}x^2}{R(x)} \left(D^{(2)} \wedge x\right) \wedge \left[\left(D^{(1)} \cdot n\right)\tilde{U} - n \wedge \left(D^{(1)} \cdot \tilde{U}\right)\right] \\
&= i\hbar \frac{\mathbf{i}x^2}{R(x)} \left[\left(D^{(2)} \cdot n\right)\left(D^{(1)} \wedge x\right) \wedge \tilde{U} - \left(D^{(1)} \wedge x\right) \wedge n \wedge \left(D^{(2)} \cdot \tilde{U}\right)\right] \\
&\quad - i\hbar \frac{\mathbf{i}x^2}{R(x)} \left[\left(D^{(1)} \cdot n\right)\left(D^{(2)} \wedge x\right) \wedge \tilde{U} - \left(D^{(2)} \wedge x\right) \wedge n \wedge \left(D^{(1)} \cdot \tilde{U}\right)\right].
\end{aligned} \tag{205}$$

As in $N = 4$, the extra terms vanish for three of the O(3) or O(2,1) generators — the three are determined by the conditions

$$D^{(1)} \cdot \tilde{U} = D^{(2)} \cdot \tilde{U} = 0 \qquad \Rightarrow \qquad D^{(1,2)} = D^{(1,2)\mu}\mathbf{e}_\mu \quad , \quad \mu = 0,1,2,3 \tag{206}$$

and

$$D^{(1)} \cdot n = D^{(2)} \cdot n = 0. \tag{207}$$

The examples discussed in section 5.5 are recovered by the choices $n = \mathbf{e}_0$ and $n = \mathbf{e}_3$.

6 Conclusion

In the presence of the field

$$F^{\mu\nu}(x) = \frac{1}{(N-2)!}\epsilon^{\mu\nu\lambda_0\lambda_1\cdots\lambda_{N-3}}G_{\lambda_0\lambda_1\cdots\lambda_{N-3}} = \frac{1}{(N-3)!}\frac{\epsilon^{\mu\nu\lambda_0\lambda_1\cdots\lambda_{N-3}}x_{\lambda_0}U_{\lambda_1\cdots\lambda_{N-3}}}{R(x)} \quad (208)$$

the extended $O(N-1,1)$ generators

$$\tilde{L}^{\mu\nu} = L^{\mu\nu} + Q^{\mu\nu} = m\left(x^\mu\dot{x}^\nu - x^\nu\dot{x}^\mu\right) + x^\mu x_\sigma F^{\sigma\nu} - x^\nu x_\sigma F^{\sigma\mu} - x^\sigma x_\sigma F^{\mu\nu} \quad (209)$$

are constructed in such a way that the commutation relations of the generators with position x^μ and velocity $\dot{x}^\mu$ satisfy the Lorentz algebra

$$\left[x^\mu,\, \tilde{L}^{\rho\lambda}\right] = i\hbar\left(x^\lambda g^{\mu\rho} - x^\rho g^{\mu\lambda}\right) \qquad \left[\dot{x}^\mu,\, \tilde{L}^{\rho\lambda}\right] = i\hbar\left(g^{\mu\rho}\dot{x}^\lambda - g^{\mu\lambda}\dot{x}^\rho\right) \quad (210)$$

when the dynamical evolution is restricted to

$$x(\tau) \in x^U = \left\{x \mid x^{\lambda_1}U_{\lambda_1\lambda_2\cdots\lambda_{N-3}} = 0\right\} \quad (211)$$

and the radial function in N dimensions has the form

$$R(x) = \left(x^2\right)^{3/2}. \quad (212)$$

Similarly, the commutation relations among the generators

$$\left[\tilde{L}^{\mu\nu},\, \tilde{L}^{\lambda\rho}\right] = i\hbar\left\{g^{\mu\lambda}\tilde{L}^{\nu\rho} - g^{\mu\rho}\tilde{L}^{\nu\lambda} - g^{\nu\lambda}\tilde{L}^{\mu\rho} + g^{\nu\rho}\tilde{L}^{\mu\lambda}\right\} + \Delta_2^{\mu\nu\lambda\rho} \quad (213)$$

with

$$\Delta_2^{\mu\nu\sigma\rho} = i\hbar\frac{x^2}{R(x)}\frac{1}{(N-3)!}\epsilon^{\mu\sigma\rho\zeta\lambda_2\cdots\lambda_{N-3}}x_\zeta g^{\nu\lambda_1}U_{\lambda_1\lambda_2\cdots\lambda_{N-3}} \quad (214)$$

satisfy the closed Lie algebra for the three rotations of O(3) or the two boosts and one rotation of O(2,1) that leave the subspace x^U invariant. By appropriate choice of U it is possible to recover either four-dimensional solution in any number of dimensions. In particular, the O(3)-invariant solution recovers the nonrelativistic case for any N, and the solution can be interpreted as a generalization of the Dirac monopole to $N > 4$. The field strength in the O(2,1)-invariant solution is associated with a potential of the type

$$A(x) \sim \left(-t^2 + \mathbf{x}^2\right)^{-1/2} \quad (215)$$

which may be seen as a relativistic generalization of the nonrelativistic Coulomb potential. A solution to the relativistic central force problems, including the bound state problem for the scalar hydrogen atom, was found [11] in the Horwitz-Piron formalism [12]

$$i\partial_\tau\psi(x,\tau) = \left[\frac{p^\mu p_\mu}{2M} + V(x)\right]\psi(x,\tau) \quad (216)$$

by making the replacement

$$r=\sqrt{(\mathbf{x}_1-\mathbf{x}_2)^2} \qquad \longrightarrow \qquad \rho=\sqrt{(\mathbf{x}_1-\mathbf{x}_2)^2-(t_1-t_2)^2} \tag{217}$$

in the argument of the classical potential $V(x)$. The discrete spectrum for the hydrogen atom was found using a potential of the type (215) when the support is restricted to the O(2,1) invariant subspace

$$\mathrm{RMS}\,(\hat{n})=\left\{x \mid \hat{n}^2>0, \quad (x_\perp)^2=[x-(x\cdot\hat{n})\hat{n}]^2>0\right\}. \tag{218}$$

The bound state was initally obtained by taking $\hat{n}=\mathbf{e}_3$ with

$$\mathrm{RMS}\,(\mathbf{e}_3)=\left\{x=\left(x^0,x^1,x^2,x^3\right) \mid (x_\perp)^2=-\left(x^0\right)^2+\left(x^1\right)^2+\left(x^2\right)^2>0\right\}, \tag{219}$$

which can be interpreted as a relaxation of condition (187). It was argued in [15] that the scalar action-at-a-distance potential is obtained, in analogy to the nonrelativistic Coulomb case, as the approximation

$$-e_0A^5(\tau,x)\to V(x)=-eA^5(x) \qquad A^\mu(\tau,x)=0 \tag{220}$$

in the 5-dimensional locally gauge invariant Schrodinger equation

$$(i\partial_\tau+e_0A_5)\psi(x,\tau)=\frac{1}{2M}(p^\mu-e_0A^\mu)(p_\mu-e_0A_\mu)\psi(x,\tau), \tag{221}$$

the quantum analog of the classical Stueckelberg mechanics given in (16). Writing the approximation (220) for the Coulomb problem as

$$A(x)=A^\alpha(x)\,\mathbf{e}_\alpha=A^5(x)\,\mathbf{e}_5=\frac{\mathbf{e}_5}{\left(-t^2+\mathbf{x}^2\right)^{1/2}} \tag{222}$$

a field of the type $G(x)$ can be found from

$$G=\partial\wedge[A\wedge\hat{n}]=\partial\wedge\left[\frac{\mathbf{e}_5\wedge\hat{n}}{\left(x^2\right)^{1/2}}\right]=-\frac{x\wedge\mathbf{e}_5\wedge\hat{n}}{\left(x^2\right)^{3/2}} \tag{223}$$

subject to the conditions

$$x^2=x^\mu x_\mu=-t^2+\mathbf{x}^2>0 \qquad x\cdot\hat{n}=0 \tag{224}$$

and it follows that

$$F=\mathbf{i}G=-\mathbf{i}_4\mathbf{e}_5\frac{x\wedge\mathbf{e}_5\wedge\hat{n}}{\left(x^2\right)^{3/2}}=\mathbf{i}_4\frac{x\wedge\hat{n}}{\left(x^2\right)^{3/2}} \tag{225}$$

In particular, taking $\hat{n} = \mathbf{e}_3$ leads to

$$G(x) = -\frac{x \wedge \mathbf{e}_3 \wedge \mathbf{e}_5}{\left(-\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2\right)^{3/2}} \tag{226}$$

$$F(x) = \mathbf{i}_4 \frac{x \wedge \mathbf{e}_3}{\left(-\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2\right)^{3/2}} \tag{227}$$

which are precisely the solution given in (185) and (186). Therefore, since this field solution emerges as a consequence of $O(N-1,1)$ symmetry, and is not dependent on the inhomogeneous Maxwell equation or the structure of the Green's function in any particular dimension N, it may be argued that the covariant generalization of the Coulomb action-at-a-distance interaction has a similar origin.

References

[1] F. J. Dyson, *Am. J. Phys.* **58** 209 (1990).

[2] N. Dombey, *Am. J. Phys.* **59** 1 85 (1991).
R.W. Brehme, *Am. J. Phys.* **59** 1 85 (1991).
J.L. Anderson, *Am. J. Phys.* **59** 1 86 (1991).
I.E. Farquhar, *Am. J. Phys.* **59** 1 87 (1991).
A. Vaidya and C. Farina, *Phys. Lett.* **153A** 265 (1991).
R. J. Hughes, *Am. J. Phys.* **60** 301 (1992).

[3] Shogo Tanimura, *Ann. Phys.* **220** 229 (1992).

[4] M. C. Land, N. Shnerb, L. P. Horwitz, *J. Math. Phys.* **36** 3263 (1995).

[5] E. C. G. Stueckelberg, *Helv. Phys. Acta* **14** 322 (1941);
Helv. Phys. Acta **14** 588 (1941).

[6] D. Saad, L. P. Horwitz and R. I. Arshansky, *Found. Phys.* **19** 1126 (1989).

[7] S. A. Hojman and L. C. Shepley, *J. Math. Phys.* **32** 142 (1991).

[8] R. M. Santilli, *Foundations of Theoretical Mechanics I*, (Springer), Berlin, 1990.

[9] A. Bérard, J. Lages, and H. Mohrbach, "Restoration of angular Lie algebra symmetries from a covariant Hamiltonian," arXiv:gr-qc/0110005;
Eur. Phys. J. **C35** 373 (2004).

Alain Bérard, Y. Grandati, and Hervé Mohrbach, "Dirac monopole with Feynman brackets," arXiv:physics/0004008.

[10] M. C. Land, *Found. Phys.* **35** 1245 (2005).

[11] L. P. Horwitz and R. Arshansky, *J. Math. Phys.* **30** 66 (1989); *J. Math. Phys.* **30** 380 (1989).

[12] L. P. Horwitz and C. Piron, *Helv. Phys. Acta* **48** 316 (1973).

[13] David Hestenes, *Space-Time Algebra*, (Gordon and Breach, New York, 1966).

[14] Pavsic, Matej, *The Landscape of Theoretical Physics: A Global View*, (Kluwer Academic, 2001).

[15] M. C. Land, *Found. Phys.* **31** 967 (2001).