Off-Shell Phenomena in Coulomb Scattering

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Abstract

In the framework of off-shell quantum electrodynamics — the quantum field theory of a covariant symplectic mechanics, in which events evolve according to a Poincaré-invariant parameter τ — we study the low energy scattering of identical scalar particles. It is shown that exchange of mass is permitted in the formalism, and we calculate scattering cross-sections for this case. In these cross-sections, the usual forward pole of standard scalar QED splits into two poles and a zero, slightly offset from the forward direction. As mass exchange vanishes, a pole-zero pair cancel, the remaining pole moves to $\theta = 0$, and the standard cross-section is recovered.

1 Introduction

Off-shell electrodynamics [1, 2] is the local gauge theory of a covariant quantum mechanics based on a dynamical theory of spacetime events. The underlying framework possesses a symplectic structure, in which a manifestly covariant Hamiltonian generates evolution through a Poincaré-invariant parameter τ. The theory provides a framework for relativistic potential theory — the treatment of many particles with mutual interaction. Within this framework, manifestly covariant generalizations have been given for the central force problem, including potential scattering [3] and the bound state [4, 5] with radiative transitions [6] and Zeeman spectra [7]. A covariant statistical mechanics [8] has also been constructed. In this covariant mechanics, a generalization of the "proper time formalism" [9 – 14], one may introduce Poincaré-invariant two-body potentials defined on an unconstrained eight-dimensional phase space, and the formulation of a many-body theory becomes accessible. The evolution parameter is not identical with the proper time of the classical motion; the particle mass is a dynamical quantity [1, 15].

In the local gauge theory, the two-body potentials are replaced by five τ-dependent gauge compensation fields, which guarantee invariance under gauge transformations of the form

$$\psi(x,\tau) \rightarrow \exp\left[ie_0\Lambda(x,\tau)\right]\ \psi(x,\tau)\ . \tag{1}$$

The invariant action [2, 16]

$$\mathrm{S} = \int d^4x d\tau \left\{\psi^*(i\partial_\tau + e_0 a_5)\psi - \frac{1}{2M}\psi^*(-i\partial_\mu - e_0 a_\mu)(-i\partial^\mu - e_0 a^\mu)\psi - \frac{\lambda}{4} f_{\alpha\beta} f^{\alpha\beta}\right\}\ , \tag{2}$$

is defined on the scalar fields $\psi(x,\tau)$ and the five gauge fields, which transform under (1) as

$$a_\mu(x,\tau) \rightarrow a_\mu(x,\tau) + \partial_\mu\Lambda(x,\tau) \qquad a_5(x,\tau) \rightarrow a_5(x,\tau) + \partial_\tau\Lambda(x,\tau)\ . \tag{3}$$

The gauge invariant kinetic term for the gauge field is $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$,where

$$\mu,\nu = 0,1,2,3 \qquad \text{and} \qquad \alpha,\beta,\gamma = 0,1,2,3,5 \tag{4}$$

and the metric is

$$g^{\alpha\beta} = \mathrm{diag}(-1,1,1,1,\sigma)\ . \tag{5}$$

The five-dimensional symmetry of $f_{\alpha\beta} f^{\alpha\beta}$, governing the homogeneous field equations, is O(4,1) or O(3,2) for $\sigma = \pm 1$. The conserved current

$$\partial_\mu j^\mu + \partial_\tau j^5 = 0 \tag{6}$$

where

$$j^5 \equiv \rho = \left|\psi(x,\tau)\right|^2 \qquad j^\mu = \frac{-i}{2M}\left\{\psi^*(\partial^\mu - ie_0 a^\mu)\psi - \psi(\partial^\mu + ie_0 a^\mu)\psi^*\right\}\ , \tag{7}$$

breaks the physical symmetry of the interacting theory to O(3,1) (the kinetic term for the matter field is linear in ∂_τ). Nevertheless, the higher symmetry is reflected in the causal properties of the wave equation

$$\partial_\alpha\partial^\alpha f^{\beta\gamma} = (\partial_\mu\partial^\mu + \partial_\tau\partial^\tau) f^{\beta\gamma} = (\partial_\mu\partial^\mu + \sigma\ \partial_\tau^2) f^{\beta\gamma} = -e(\partial^\beta j^\gamma - \partial^\gamma j^\beta)\ , \tag{8}$$

which follows from the gauge field equations

$$\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda} j^\alpha = e j^\alpha \qquad\qquad \epsilon^{\alpha\beta\gamma\delta\epsilon}\partial_\alpha f_{\beta\gamma} = 0 \tag{9}$$

where j^α is given by (7). From the dimensional constants, λ and e_0, one identifies $e_0/\lambda = e$ as the dimensionless Maxwell charge[1]. It follows from (6) that $\left|\psi(x,\tau)\right|^2$ is a probability density in the first quantized theory. The matter field equation

$$i\partial_\tau\ \psi(x,\tau) = \left[\frac{1}{2M}(-i\partial^\mu - e_0 a^\mu)(-i\partial_\mu - e_0 a_\mu) - e_0 a_5\right]\ \psi(x,\tau) \tag{10}$$

defines evolution of the event amplitude $\psi(x,\tau)$ through instantaneous interaction with the $(4\oplus 1)$-field a_α.

2 Quantization and Perturbation Theory

The canonical and path integral quantization of off-shell electromagnetism has been given in [16, 17, 18]. The underlying symplectic structure provides a natural setting for the Jackiw-Fadeev quantization procedure [19], by which the constraint equations resulting from gauge invariance are solved and the constrained degrees of freedom are eliminated from the Lagrangian. We will briefly summarize the extension to the path integral. To make the action linear in τ-derivatives, we introduce the notation $\epsilon^\mu = f^{5\mu}$ and the "first order form"

$$f_{\alpha\beta}f^{\alpha\beta} = f_{\mu\nu}f^{\mu\nu} + 2f^{5\mu}f_{5\mu} = f_{\mu\nu}f^{\mu\nu} + 2\sigma\left[2(\sigma\partial_\tau a^\mu - \partial^\mu a^5)\epsilon_\mu - \epsilon^\mu\epsilon_\mu\right]\ . \tag{11}$$

Integrating by parts and collecting terms in a_5, the action becomes

$$\begin{aligned} \mathrm{S} \ = \ \int d^4x d\tau \Big[& i\psi^*\dot{\psi} - \lambda\epsilon_\mu\dot{a}^\mu - \frac{1}{2M}\psi^*(-i\partial_\mu - e_0 a_\mu)(-i\partial^\mu - e_0 a^\mu)\psi \\ & - \frac{\lambda}{4}f_{\mu\nu}f^{\mu\nu} + \frac{\lambda\sigma}{2}\epsilon^\mu\epsilon_\mu + a_5(e_0\psi^*\psi - \lambda\partial^\mu\epsilon_\mu)\Big]\ . \end{aligned} \tag{12}$$

In the path integral,

$$\mathcal{Z} = \frac{1}{\mathcal{N}}\int \mathcal{D}\psi^*\ \mathcal{D}\psi\ \mathcal{D}a_\mu\ \mathcal{D}a_5\ \mathcal{D}\epsilon_\mu e^{iS}\ , \tag{13}$$

[1] The τ-integral of (6) reduces to $\partial_\mu J^\mu = 0$ for $J^\mu = \int d\tau\ j^\mu$, from which one identifies J^μ as the Maxwell field. The τ-integral of the spacetime part of the first of (9) then reduces to the inhomogeneous Maxwell equation. See [2, 6, 7] for further discussion.

integration over a_5 places $\delta(e_0\psi^*\psi - \lambda\partial^\mu\epsilon_\mu)$ into the measure, giving the constraint

$$e_0\psi^*\psi - \lambda\partial^\mu\epsilon_\mu = 0 \quad \Rightarrow \quad \partial^\mu\epsilon_\mu = \frac{e_0}{\lambda}\psi^*\psi = e\rho \quad \Rightarrow \quad \partial_\alpha f^{5\alpha} = ej^5 \ , \tag{14}$$

the Gauss law of the theory, as seen from (9). The constraint equation (14) can be solved through the decomposition

$$\epsilon^\mu = (\epsilon_\perp)^\mu + e\partial^\mu \int d^4y \ \delta\left((x-y)^2\right) \ \rho(y,\tau) \qquad \text{where} \qquad \partial_\mu(\epsilon_\perp)^\mu = 0 \ . \tag{15}$$

Performing a similar decomposition of a^μ,

$$a^\mu = (a_\perp)^\mu + \partial^\mu \int d^4y \ \delta\left((x-y)^2\right) \ \Lambda(y) \qquad \text{where} \qquad \partial_\mu(a_\perp)^\mu = 0 \tag{16}$$

(by which we implicitly choose the gauge condition $\partial_\mu a^\mu = \Lambda$, where Λ is τ-independent[2]), and integrating over $\mathcal{D}\epsilon_{||}$ and $\mathcal{D}a_{||}$, only unconstrained degrees of freedom remain in the action. The Darboux transformation

$$\psi \longrightarrow \exp\left[ie_0 \ \left[G\Lambda\right](x)\right] \ \psi \ , \tag{17}$$

in which $\left[G\Lambda\right](x)$ is shorthand for

$$\left[G\Lambda\right](x) = \int d^4y \ \delta\left((x-y)^2\right) \ \Lambda(y) \ , \tag{18}$$

leaves the action diagonal and unconstrained. Integrating over $\epsilon_\perp$ — the conjugate momentum for $a_\perp$ — puts the path integral into the form

$$\mathcal{Z} = \frac{1}{\mathcal{N}} \int \mathcal{D}\psi^* \ \mathcal{D}\psi \ \mathcal{D}(a_\perp)_\mu \ e^{iS} \tag{19}$$

where the action S is given by (see also [17])

$$\begin{aligned} \mathrm{S} \ = \ \int d^4x d\tau \Big\{ & i\psi^*\dot{\psi} - \frac{1}{2M}\psi^*(-i\partial_\mu - e_0(a_\perp)_\mu)(-i\partial^\mu - e_0(a_\perp)^\mu)\psi \\ & + \frac{\lambda}{2}(a_\perp)_\mu[\Box + \sigma\partial_\tau^2](a_\perp)^\mu - \frac{\lambda\sigma}{2}e^2\rho[G\rho]\Big\} \ . \end{aligned} \tag{20}$$

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[2]The decompositions (15) and (16) can be achieved by performing a gauge transformation

$$a^\mu = \hat{a}^\mu + \partial^\mu\phi$$

such that

$$\partial_\tau\partial_\mu\hat{a}^\mu = 0 \qquad \text{and} \qquad -\partial_\mu\partial^\mu\hat{a}^5 = \rho \ ,$$

so that $\partial_\mu\partial^\mu\phi + \partial_\mu\hat{a}^\mu = W(x)$,where $W(x)$ is τ-independent. It follows that $\partial_\tau\Lambda = 0$, for $\Lambda = \partial_\mu\hat{a}^\mu$. The off-light cone contributions to the five space Green's function then vanish [21].

where we have expanded $(f_\perp)_{\mu\nu}(f_\perp)^{\mu\nu}$ and used the transversality of $(a_\perp)_\mu$. We recognize the "inverse propagators" $[\Box + \sigma\partial_\tau^2]$ for the free gauge field and $\left[i\partial_\tau + \frac{1}{2M}\Box\right]$ for the free matter field. From the interaction part,

$$\mathrm{S}_{\mathrm{int}} = \int d^4x d\tau \left\{ -\frac{ie_0}{2M} a_\mu(\psi^*\partial^\mu\psi - \psi\partial^\mu\psi^*) - \frac{e_0^2}{2M} a_\mu a^\mu |\psi|^2 - \frac{\lambda\sigma}{2} e^2 \rho[G\rho] \right\} \tag{21}$$

we discard the last term, which has the form of a c-number energy density representing the mass-energy equivalent required to assemble the matter field[3], and read-off the Feynman rules:

matter propagator	directed line	$\dfrac{1}{(2\pi)^5} \dfrac{-i}{\frac{1}{2M}p^2 - P - i\epsilon}$
photon propagator	photon line	$\dfrac{1}{(2\pi)^5} \dfrac{1}{\lambda} \mathcal{P}^{\mu\nu} \dfrac{-i}{k^2 + \sigma\kappa^2 - i\epsilon}$
three-particle interaction	vertex factor	$\frac{e_0}{2M} i(p+p')^\nu \; (2\pi)^5 \delta^4(p - p' - k)\delta(P - P' + \sigma\kappa)$
four-particle interaction	vertex factor	$\dfrac{-ie_0^2}{M}(2\pi)^5 g_{\mu\nu}\delta^4(k - k' - p' + p)\delta(\sigma\kappa - \sigma\kappa' + P' - P)$

The Feynman rules for the S-matrix elements, follow from the LSZ reduction formulas derived in [20], which for the scalar field, requires that incoming and outgoing propagators be replaced by 1. It was seen in [20] that the Ward identity for off-shell quantum electrodynamics connects the three-particle and four-particle vertices. It was also shown that while no matter loops exist in the theory (due to the retarded propagation), renormalization requires a cut-off in photon loops (a natural formulation of this cut-off was suggested in [22]).

[3]The coupling λe^2 is characteristic of the classical Lorentz force due to a charge distribution [16].

3 The Scattering Process $\mathrm{B}+\mathrm{T}\to 1+2$

We treat the case of elastic scattering of two identical particles of mass m, in relative coordinates:

$$P = \frac{1}{2}(p_{\mathrm{T}}+p_{\mathrm{B}}) \qquad p = p_{\mathrm{T}} - p_{\mathrm{B}} \tag{22}$$

$$P' = \frac{1}{2}(p_1+p_2) \qquad p' = p_1 - p_2 \tag{23}$$

In the center of mass system, $\vec{p}_{\mathrm{B}}+\vec{p}_{\mathrm{T}}=\vec{p}_1+\vec{p}_2=0$, so that $P=\left(\frac{1}{2}\sqrt{s},\vec{0}\right)$, where the usual Mandelstam parameter in this metric is $s=-(p_{\mathrm{B}}+p_{\mathrm{T}})^2$. Similarly,

$$p' = p_1 - p_2 = (E(p_1)-E(p_2), 2\vec{p}_1) \tag{24}$$

is spacelike, and we introduce the parameterization

$$p' = \rho\,(\sinh\beta, \cosh\beta\hat{n}) \ . \tag{25}$$

From the δ-functions in the Feynman rules above, it is clear that the total four-momentum and total mass are conserved, but the individual particle masses are not. It follows that the conserved quantities in relative coordinates are

$$P = P' \qquad \text{and} \qquad p^2 = (p')^2 = \rho^2 = s - 2(m_1^2+m_2^2) \ . \tag{26}$$

The exchange in mass is characterized by

$$\Delta m^2 = (m_1)^2 - (m_2)^2 = (p_2)^2 - (p_1)^2 = \sqrt{s}\ \rho\sinh\beta \ . \tag{27}$$

The cross-section is given by [20]

$$\frac{d\sigma^{(3)}}{d\Omega\ dt_{\mathrm{int}}} = \frac{1}{64\pi^2}\frac{|\vec{p}_f|}{s\,|\vec{p}_i|}|\langle p_1p_2|\tilde{\mathcal{T}}|p_{\mathrm{T}}p_{\mathrm{B}}\rangle|^2 \ , \tag{28}$$

where $\mathcal{T}=(\lambda/4M^2)^2\,\tilde{\mathcal{T}}$ is the usual transition matrix, and $|\vec{p}_f|$ is β-dependent. The cross-section $d\sigma^{(3)}$ has dimensions of volume [3], which is compensated by the time dt_{int} which characterizes the change in time-synchronization associated with the mass exchange [20].

Using the Feynman rules above we find the transition matrix

$$\langle 1\ 2|\mathcal{T}|\mathrm{T\ B}\rangle = \frac{e_0 e}{(2M)^2}\left\{\frac{s-u-\frac{(p_{\mathrm{T}}^2-p_1^2)^2}{t}}{t-\sigma(\kappa_{p_{\mathrm{T}}}-\kappa_{p_1})^2}+\frac{s-t-\frac{(p_{\mathrm{T}}^2-p_2^2)^2}{u}}{u-\sigma(\kappa_{p_{\mathrm{T}}}-\kappa_{p_2})^2}\right\}\ . \tag{29}$$

where the Mandelstam parameters

$$t=-(p_{\mathrm{T}}-p_1)^2=-\frac{1}{4}(p-p')^2 \qquad\qquad u=-(p_{\mathrm{T}}-p_2)^2=-\frac{1}{4}(p+p')^2 \tag{30}$$

are

$$t=-2|\vec{p}_{\mathrm{T}}|^2(1-\cosh\beta\ \cos\theta) \qquad \text{and} \qquad u=-2|\vec{p}_{\mathrm{T}}|^2(1+\cosh\beta\ \cos\theta)\ . \tag{31}$$

Notice that (29) — (31) agree with the standard on-shell expressions when $\beta=0$. Introducing $\xi=\Delta m^2/M^2$ the amplitude becomes

$$\langle 3\ 4|\mathcal{T}|1\ 2\rangle = \frac{e_0 e}{(2M)^2}\left\{\frac{1}{t}\,\frac{4st-4ut-M^4\xi^2}{4t-\frac{\sigma}{4}M^2\xi^2}+\frac{1}{u}\,\frac{4su-4tu-M^4\xi^2}{4u-\frac{\sigma}{4}M^2\xi^2}\right\}\ , \tag{32}$$

which has two forward poles, one at

$$t=0 \qquad \Rightarrow \qquad \cos\theta=\frac{1}{\cosh\beta}=\left(1+\frac{M^2\xi}{\sqrt{s}\rho}\right)^{-1/2} \tag{33}$$

and one at

$$t-\frac{\sigma}{16}M^2\xi^2=0 \qquad \Rightarrow \qquad \cos\theta=\frac{1}{\cosh\beta}\left[1+\frac{\sigma s}{32M^2}\sinh^2\beta\right]\ . \tag{34}$$

The forward part has zeros, at

$$\cos\theta=\frac{-s\pm\sqrt{(s+\rho^2)+M^4\xi^2}}{\rho^2\cosh\beta}\ . \tag{35}$$

For very small ξ, (35) becomes

$$\cos\theta\simeq\left(1-\frac{M^2\xi}{2\sqrt{s}\rho}\right)\left[1+\frac{M^4\xi^2}{2(s+\rho^2)}\right]\approx\left(1-\frac{M^2\xi}{2\sqrt{s}\rho}\right)\ . \tag{36}$$

In the on-shell limit, when $\xi\to 0$, this zero cancels one of the poles, restoring the usual forward pole. The symmetry around $\theta=\pi/2$ is preserved in the off-shell result. Away from the forward and reverse directions, the cross-section is very close to the standard Klein-Gordon expression.

In Figure 1, we show the cross-section for $\theta = 0$ to $90°$ for $m = 0.5$ MeV and $\sqrt{s} = 1.2$ MeV (the off-shell phenomena are most pronounced at low energies for which the particle masses are significant), and $\xi = 0$, 0.01, and 0.02. In Figure 2, we present a detail of the pole-zero structure of the $\xi = 0.01$ case and in Figure 3, we present a detail of the pole-zero structure of the $\xi = 0.02$ case. We choose a mass value close to the electron mass, since data on elastic electron-electron scattering is readily available — the effects of spin are negligible at the relevant angles ($\theta \sim 3°$- $4°$). We emphasize that in this theory, mass exchange is an essentially kinematic phenomenon, since the hyperangle β is on the same footing as the spacial angles θ, ϕ. We remark that the *forward elastic* cross-section is obtained by the condition $\theta = 0$ and $\beta = 0$ (i.e., identical in-state and out-state [23]), and as remarked above, the condition $\beta = 0$ restores the standard Klein-Gordon cross-section. Therefore, the forward cross-section derived from (32) for $\theta = 0$ and $\beta \neq 0$ does not constrain the total cross-section through the optical theorem.

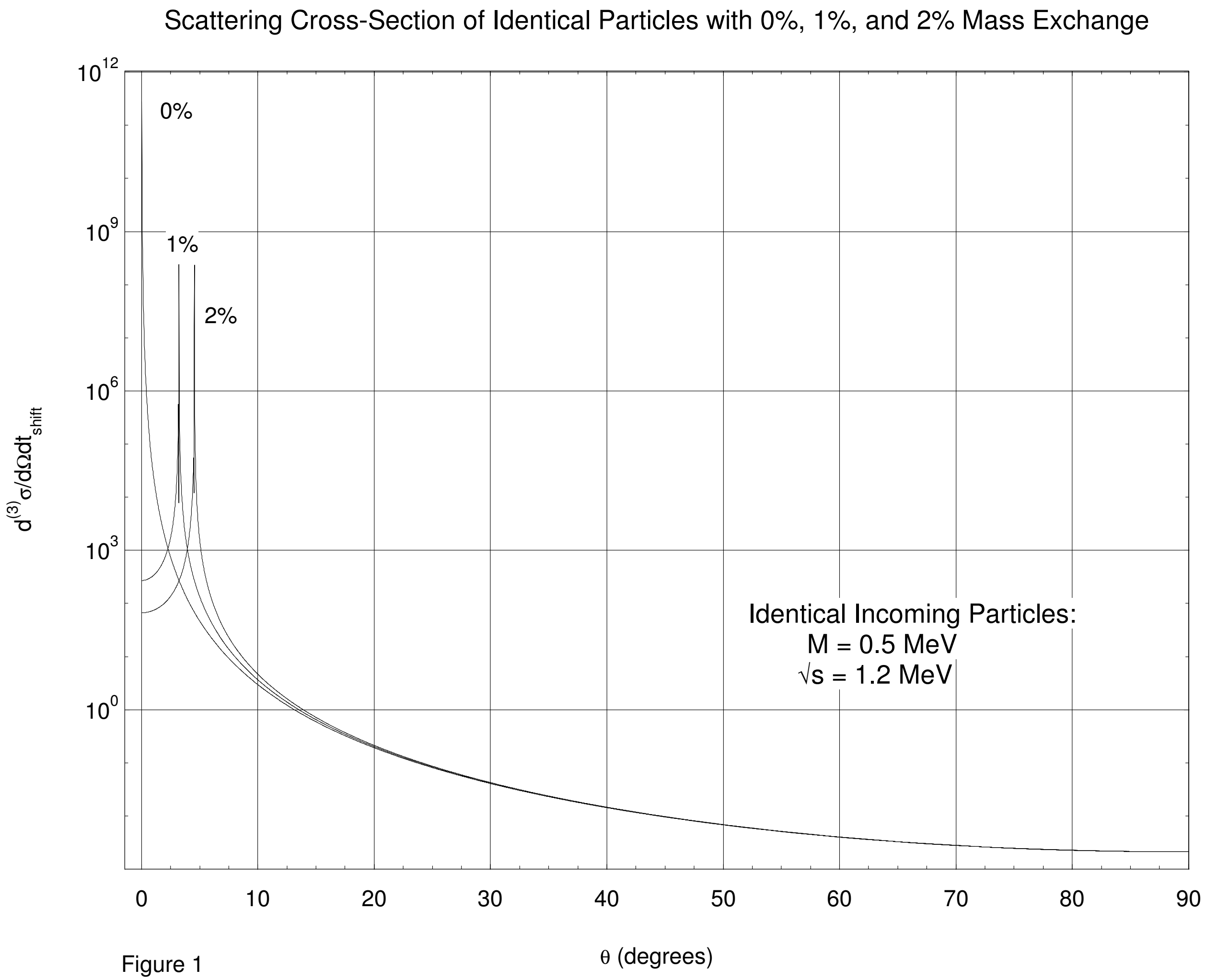
Scattering Cross-Section of Identical Particles with 0%, 1%, and 2% Mass Exchange
10^{12}
10^{9}
10^{6}
10^{3}
10^{0}
$d^{(3)}\sigma/d\Omega dt_{shift}$
0%
1%
2%
Identical Incoming Particles:
M = 0.5 MeV
√s = 1.2 MeV
0
10
20
30
40
50
60
70
80
90
θ (degrees)

Figure 1

Detail of Pole-Zero Structure in Scattering Cross-Section with 1% Mass Exchange

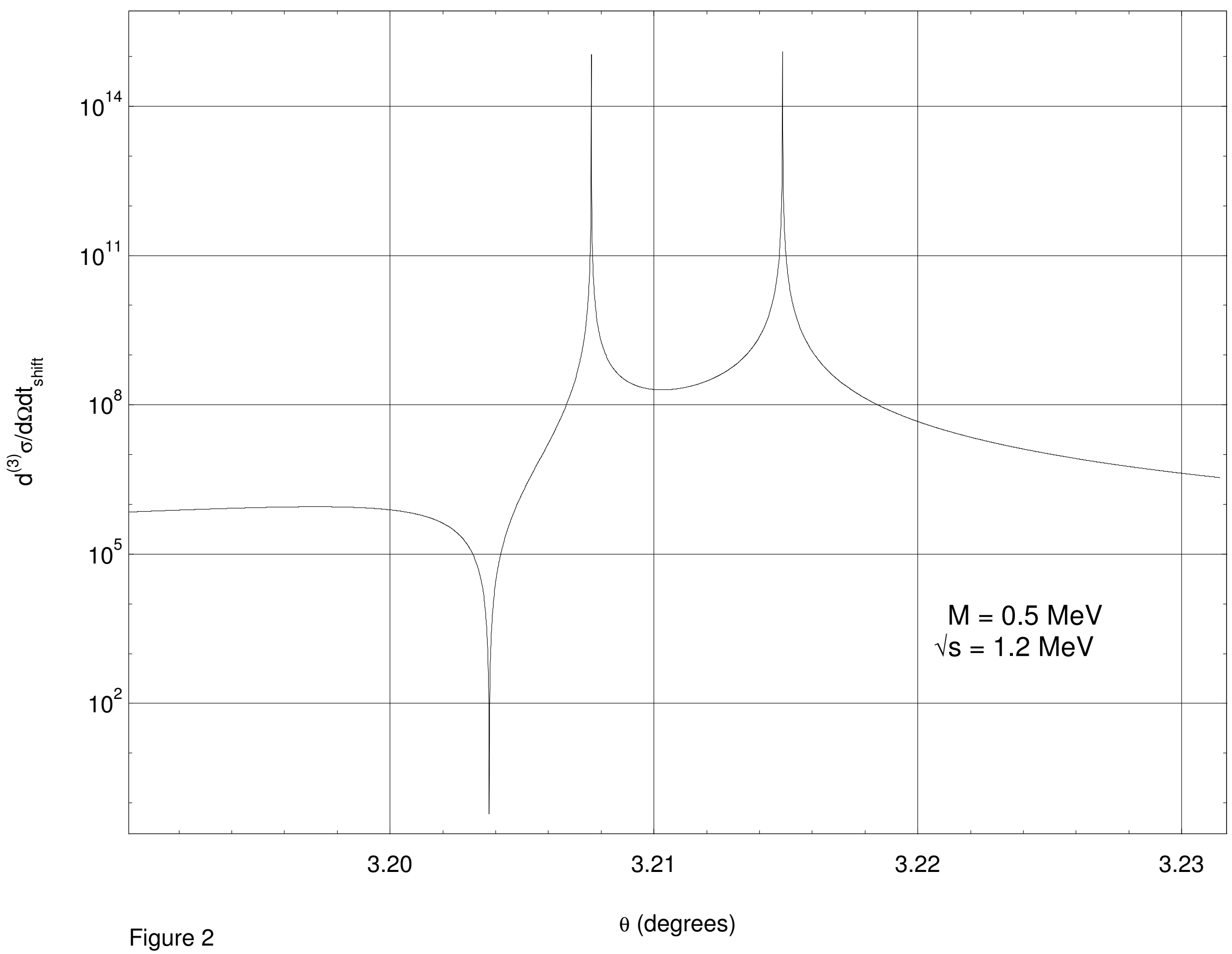

Figure 2

Detail of Pole-Zero Structure in Scattering Cross-Section with 2% Mass Exchange

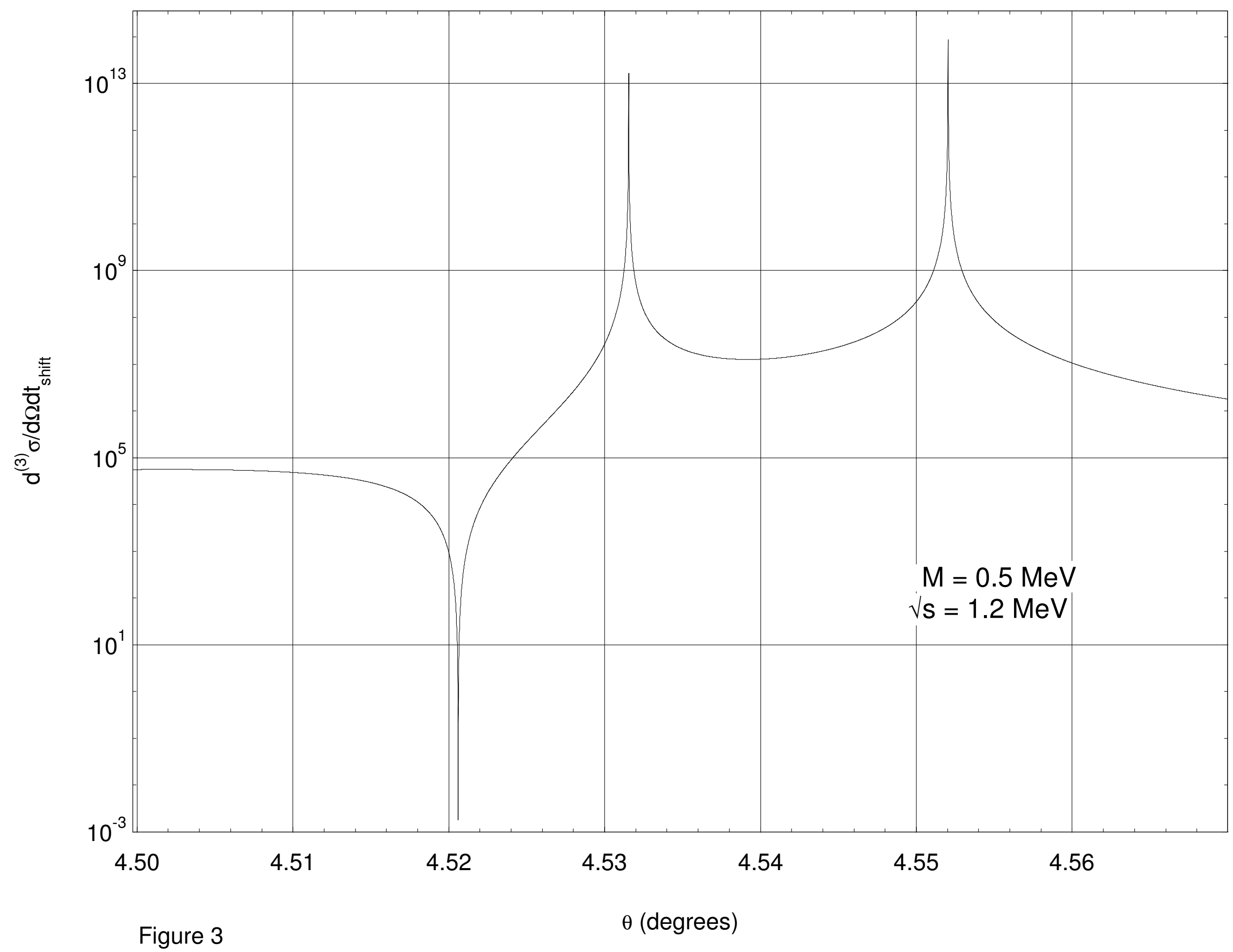

Figure 3

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