The Classical Coulomb Problem in Pre-Maxwell Electrodynamics

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Abstract

We explore certain difficulties in the covariant classical mechanics associated with off-shell electrodynamics, through an examination of the classical Coulomb problem. We present a straightforward solution of the classical equations of motion, for a test event traversing the field induced by a "fixed" event (an event moving uniformly along the time axis at a fixed point in space). This solution reveals the essential difficulties in the formalism at the classical level. We then offer a new model of the particle, as a certain distribution of events on the worldline, which eliminates these difficulties and permits comparison of classical off-shell electrodynamics with the standard Maxwell theory. In this model, the "fixed" event induces a Yukawa-type potential, permitting a semi-classical identification of the pre-Maxwell time scale λ with the inverse mass of the intervening photon. Numerical solutions to the equations of motion are compared with the standard Maxwell solutions — they are seen to coincide when $\lambda > 10^{-6}$ seconds, providing an initial estimate of this parameter.

1 Introduction

The classical mechanics associated with the pre-Maxwell electromagnetic theory is defined by the classical Lagrangian [1, 2, 3]

$$L = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu + \frac{e_0}{c} \dot{x}^\alpha a_\alpha - \frac{\lambda}{4c} f^{\alpha\beta} f_{\alpha\beta} \ . \tag{1}$$

where

$$x^5 = c\tau \qquad\qquad \partial_5 = \frac{1}{c}\partial_\tau \qquad\qquad j^5 = c\rho \ , \tag{2}$$

for

$$\lambda, \mu, \nu = 0, 1, 2, 3 \qquad\qquad \text{and} \qquad\qquad \alpha, \beta, \gamma = 0, 1, 2, 3, 5 \tag{3}$$

The gauge invariant quantity $f_{\alpha\beta}$ is defined by $f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha$ .The classical Lorentz force [2] found by variation of (1) with respect to x^μ, is

$$M\, \ddot{x}^\mu = \frac{e_0}{c}\, f^\mu{}_\alpha(x,\tau)\, \dot{x}^\alpha \qquad\qquad \frac{d}{d\tau}(-\frac{1}{2} M \dot{x}^2) = e_0\ f_{5\alpha} \dot{x}^\alpha \ . \tag{4}$$

Since particle mass is not separately conserved, pair annihilation is classically permitted.

In formally raising the index $\beta = 5$ in $f^{\mu 5} = \partial^\mu a^5 - \partial^5 a^\mu$, Sa'ad et. al. argue that the action suggests a higher symmetry containing O(3,1) as a subgroup, that is, either O(4,1) or O(3,2), with metric

$$g^{\alpha\beta} = \text{diag}(-1,1,1,1,\sigma) \qquad\qquad \sigma = \pm 1, \tag{5}$$

Variation of (1) with respect to a_α yields

$$\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda c} j^\alpha = \frac{e}{c}\, j^\alpha \qquad\qquad \epsilon^{\alpha\beta\gamma\delta\epsilon}\partial_\alpha f_{\beta\gamma} = 0 \tag{6}$$

where e_0/λ is identified as the dimensionless charge e, and the current j^α associated with an event $X^\alpha = \big(X^\mu(\tau), c\tau\big)$ is given by

$$j^\alpha(x,\tau) = c\,\frac{dX^\alpha}{d\tau}\delta^4\Big(x^\mu - X^\mu(\tau)\Big)\ . \tag{7}$$

The current conservation law may be written as $\partial_\alpha j^\alpha = 0$.

Under the boundary conditions $f^{5\mu} \to 0$, pointwise as $\tau \to \pm\infty$, we recover Maxwell's equations as

$$\partial_\nu F^{\mu\nu} = \frac{e}{c} J^\mu \qquad\qquad \epsilon^{\mu\nu\rho\lambda}\partial_\mu F_{\nu\rho} = 0 \tag{8}$$

where

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau f^{\mu\nu}(x,\tau) \qquad\qquad \text{and} \qquad\qquad A^\mu(x) = \int_{-\infty}^{\infty} d\tau\ a^\mu(x,\tau)\ . \tag{9}$$

Therefore, $a^\alpha(x,\tau)$ has been called the pre-Maxwell field. Since $e_0 a_\mu$ and eA_μ must have the same dimensions, it follows from (9) that λ (and hence $e_0 = \lambda e$) must have dimensions of time. Although the parameter λ does not appear in the field equations (6), it does appear in the Lorentz force (4) through e_0. The presense of this dimensional parameter in the equations of motion is a characteristic problem in the classical formalism.

The physical Lorentz covariance of the current j^α breaks the higher symmetry of the free field equations to O(3,1). Nevertheless, the wave equation

$$\partial_\alpha\partial^\alpha\ a^\beta = (\partial_\mu\partial^\mu + \partial_\tau\partial^\tau)\ a^\beta = (\partial_\mu\partial^\mu + \sigma\ \partial_\tau^2)\ a^\beta = -\frac{e}{c}\, j^\beta\ , \tag{10}$$

reflects the causal properties of the higher symmetry through the operator on the left hand side. The classical Green's function for (10), defined through

$$\partial_\alpha \partial^\alpha G(x, x^5) = -c\delta^4(x, x^5) \ , \tag{11}$$

is given by [4]

$$G(x, x^5) = -\frac{c}{4\pi}\delta(x^2)\delta(x^5) - \frac{c}{2\pi^2}\frac{\partial}{\partial x^2}\frac{\theta(-\sigma g_{\alpha\beta}x^\alpha x^\beta)}{\sqrt{-\sigma g_{\alpha\beta}x^\alpha x^\beta}} \ . \tag{12}$$

It follows from (10) and (11) that the potential induced by a known current is given by

$$a^\beta(x, \tau) = -\frac{e}{c}\int d^4x' dx^{5\prime} \ \frac{1}{c} \ G(x - x', x^5 - x^{5\prime}) \ j^\beta(x', c\tau') \tag{13}$$

Under concatenation, the first term of (12) becomes the Maxwell Green's function $D(x) = -\frac{1}{4\pi} \ \delta(x^2)$, while the second term — which induces spacelike or timelike correlations [4], depending on the signature σ — vanishes. This concatenation guarantees that the Maxwell potential is related to the Maxwell current in the usual manner:

$$A^\mu(x) = \int d\tau \ a^\mu(x, \tau) = -\frac{e}{c}\int d^4x' D(x - x') \ J^\mu(x') \ . \tag{14}$$

Therefore, we will refer to the Maxwell and the correlation terms of the Green's function and the induced potentials.

2 The Coulomb Problem

The significant differences between the pre-Maxwell theory and standard electrodynamics raise questions about how the former may be used to describe the usual electromagnetic phenomenology. In particular, we are interested in describing the trajectory of a low energy particle in the field induced by a "static" particle. We approach this problem in the pre-Maxwell theory by studying the motion of a charged event in the field produced by a second event, moving uniformly along the time axis. As in classical Rutherford scattering, we hold the inducing event "fixed" on its time axis, neglecting the field of the scattered event, and the radiation field that it would induce. A more detailed treatment is given in [5].

According to equation (7), the current associated with an event moving uniformly along the time axis,

$$X^0(\tau) = ct = c\tau \qquad\qquad \vec{X}(\tau) = 0 \tag{15}$$

is given by

$$j^0(x,\tau) = j^5(x,\tau) = c\frac{dX^0}{d\tau}\delta(x^0 - c\tau)\delta^3(\vec{x}) = c\;\delta(t-\tau)\delta^3(\vec{x})\;, \tag{16}$$

with $\vec{j}(x,\tau) = 0$. Therefore, from (13) we will have

$$a^5(x,\tau) = a^0(x,\tau) \qquad\qquad \vec{a}(x,\tau) = 0\;, \tag{17}$$

and the only non-zero field components are

$$f^{0k} = -\partial^k a^0 \qquad f^{5k} = -\partial^k a^0 \qquad f^{05} = -\frac{1}{c}\left(\partial_0 a^0 + \sigma\partial_\tau a^0\right)\;. \tag{18}$$

The independent components of the Lorentz force are

$$M\;\ddot{x}^0 = -\lambda\frac{e}{c}\;\left[(\partial_k a^0)\dot{x}^k + \sigma\frac{1}{c}\;(\partial_0 + \sigma\partial_\tau)\;a^0\right] \tag{19}$$

and

$$M\;\ddot{x}^k = \frac{e_0}{c}\;f^{k\alpha}\dot{x}_\alpha = -\lambda\frac{e}{c}\;(\partial_k a^0)(\dot{x}^0 - \sigma c)\;, \tag{20}$$

where we used the antisymmetry of $f^{\alpha\beta}$ and $\dot{x}_5 = \sigma c$. Since at low energies, $\dot{x}^0 \sim c$, we notice that the choice of $\sigma = 1$ in (20) will rule out identification with the standard Maxwell equations of motion. We therefore choose $\sigma = -1$ and study only the case of broken O(3,2) symmetry. With this choice,

$$M\;\ddot{x}^0 = -\lambda\frac{e}{c}\;\left[(\partial_k a^0)\dot{x}^k - \frac{1}{c}\;(\partial_t - \partial_\tau)\;a^0\right] \qquad M\;\ddot{x}^k = -\lambda e\;(\partial_k a^0)(\dot{t}+1)\;. \tag{21}$$

Since λ does not appear in equations (13) or (16), the coupling of the induced field to the test event will evidently depend on this parameter.

We must now calculate the potential $a^0(x,\tau)$ from the Green's function (12) and the current (16). The Maxwell part of the potential is given by

$$a^0(x,\tau) = -\frac{e}{4\pi R}\;\frac{1}{2}\Big[\delta(t-\tau-R/c) + \delta(t-\tau+R/c)\Big]\;, \tag{22}$$

where $R = |\vec{x}|$. As required for the Maxwell part, $A^0(x) = \int d\tau a^0(x,\tau) = -\frac{e}{4\pi R}$, the concatenated potential has the form of the Coulomb potential due to a "fixed" source. It turns out that the causality properties of the correlation term in the Green's function require that $a_{\text{correlation}}$ vanish [5]. It may be shown that the correlation term will only contribute when the inducing charge undergoes acceleration.

The support of the potentials is restricted, by the delta-functions they contain, to the light cone of the source event and so the test event moving in the induced field will be free except when its trajectory satisfies $t - \tau - R/c = 0$. For $x^\mu(0) = s^\mu$ and $\dot{x}^\mu = u^\mu$, the interaction will occur at τ^*, given by $\tau^* = \frac{1}{u_x}\left(-s_x + \sqrt{s_t^2 - s_y^2}\right)$so that

$$R^* = R(\tau^*) = s_t \qquad x^* = x(\tau^*) = \sqrt{s_t^2 - s_y^2} \qquad y^* = y(\tau^*) = s_y \tag{23}$$

The Lorentz force (21) may be integrated as

$$\vec{u}(\tau^* + \epsilon) - \vec{u}(\tau^* - \epsilon) = \frac{\lambda e^2}{4\pi M}\nabla \int_{\tau^*-\epsilon}^{\tau^*+\epsilon} d\tau \frac{1}{R}\delta(t - \tau - R/c) = -\frac{\lambda e^2}{4\pi M}\frac{1}{(R^*)^2}\hat{x}^* \ . \tag{24}$$

Similarly,

$$\dot{t}(\tau^* + \epsilon) - \dot{t}(\tau^* - \epsilon) = -\frac{\lambda e^2}{Mc}\frac{1}{8\pi(R^*)^2}\ \hat{x}^* \cdot \vec{x}^* = 0 + o(v/c) \ . \tag{25}$$

Imposing conservation of energy leads to the scattering angle

$$\cot\frac{\theta}{2} = \frac{s_y}{\sqrt{s_t^2 - s_y^2}} \ . \tag{26}$$

This may be compared with the asymptotic scattering angle for non-relativistic Rutherford scattering, given by [6]

$$\cot\frac{\theta}{2} = \frac{2Es_y}{e^2/4\pi} = \frac{4\pi M u^2 s_y}{e^2} \ . \tag{27}$$

Requiring equality of (26) and (27) leads to

$$\sqrt{s_t^2 - s_y^2} = \frac{e^2}{4\pi M u^2} \ , \tag{28}$$

which fixes a particular value for s_t. The scattering angle will agree with the asymptotic value obtained in the standard Maxwell theory only if λ is equal to the distance at the time of interaction divided by the incoming speed.

The solution presented above is unsatisfactory for a number of reasons, aside from the essential discrepancy between the piecewise linear solution found here and the smooth acceleration expected in the Coulomb problem. Notice first that we will find no solution for τ^* if $s_t < s_y$, meaning that a test event with these initial conditions will pass the source event without interaction. In particular, two events moving together along the time axis, at separation R but no initial relative offset in time will experience no Coulomb force.

The dependence of the scattering angle on the initial condition s_t of the time coordinate exposes the underlying difficulty in the straightforward approach. The standard Maxwell current $J^\mu(x)$ has its support along the entire world line of the particle, while the support of the pre-Maxwell current is concentrated at one point of the world line; in the classical case, this point (event) depends explicitly on the time synchronization (initial condition) with respect to τ. In the quantum regime, when states are defined with sharp asymptotic masses, this τ-synchronization is completely undetermined because of the uncertainty relation between τ and mass [7]. Therefore, results in relativistic quantum mechanics and off-shell quantum electrodynamics do not suffer this dependence. Furthermore, Arshansky, Horwitz, and Lavie [8] have argued that measurements made at a spacetime point x^μ do not take place at a definite τ, but rather concatenate all events — occurring at various values of τ — which could contribute to the event at x^μ. From this point of view, the initial τ-synchronization of events in a scattering experiment (and so s_t) cannot be precisely measured, even at the classical level, and may be associated with a fundamental uncertainty. The success of the formalism in the quantum regime suggests that the formulation of the classical equations of motion be modified to take account of this uncertainty.

3 Particles as Distributions of Events

We wish to overcome the difficulties presented in the previous chapter, by incorporating in the description of classical particles an uncertainty in their initial conditions with respect to τ, and we take as our starting point the observation that the Maxwell current $J^\mu(x)$ determined by measurement devices [8] is insensitive to this uncertainty. To see this, we take a normalized distribution function $\varphi(\alpha)$, with $\int_{-\infty}^{\infty} d\alpha\ \varphi(\alpha) = 1$, and replace the event current $j^\mu(x,\tau)$, given in (7) for the sharply defined event $x^\mu(\tau)$, with the current

$$j_\varphi^\beta(x,\tau) = \int_{-\infty}^{\infty} d\alpha\ \varphi(\alpha)\ j^\beta(x,\tau-\alpha)\ . \tag{29}$$

Since the concatenation integral may be shifted by $\tau \longrightarrow \tau' = \tau - \alpha$, this will not affect the concatenated Maxwell current. In the particular case discussed in the previous section, the current associated with a particle moving uniformly along the time axis is given by

$$j_\varphi^0(x,\tau) = \int_{-\infty}^{\infty} d\alpha\ \varphi(\alpha)\ j^0(x,\tau-\alpha) = c\ \delta^3(\vec{x})\ \varphi(t-\tau)\ . \tag{30}$$

The pre-Maxwell potential induced by this current is then

$$a_\varphi^0(x,\tau) = -\frac{e}{4\pi R}\ \frac{1}{2}\Big[\varphi(t-\tau-R/c) + \varphi(t-\tau+R/c)\Big]\ . \tag{31}$$

A convenient choice of distribution function is

$$\varphi(\alpha) = \frac{1}{2\lambda} e^{-|\alpha|/\lambda}\ , \tag{32}$$

in which λ represents the width of the distribution. For this distribution function, the induced potential is given by

$$a_\varphi^0(x,\tau) = -\frac{e}{4\pi R}\ \frac{1}{2\lambda}\ \frac{1}{2}\ \Big[e^{-\big|t-\tau-R/c\big|/\lambda} + e^{-\big|t-\tau+R/c\big|/\lambda}\Big]\ , \tag{33}$$

and so

$$M\ \ddot{x}^0 = -\frac{1}{2}\ \frac{e}{c}\ \Big[(\partial_k \tilde{a}_\varphi^0)\dot{x}^k - \frac{1}{c}\ (\partial_t - \partial_\tau)\ \tilde{a}_\varphi^0\Big] \qquad M\ \ddot{x}^k = -e\ (\partial_k \tilde{a}_\varphi^0)\frac{\dot{t}+1}{2}\ , \tag{34}$$

where $\tilde{a}_\varphi^0 = 2\lambda\ a_\varphi^0$ now includes the factor 2λ and so has the units of A^0. In the low energy limit ($v/c \sim 0$), with $t \sim \tau$ (and $\dot{t} \sim 1$), we obtain the standard equations of motion for a particle in a classical Yukawa potential,

$$M\ \ddot{x}^0 = 0 \qquad M\ \vec{x} = e\ (-\nabla \tilde{a}_\varphi^0) = -e\ \nabla\ [-\frac{e}{4\pi R}\ e^{-R/\lambda c}]\ . \tag{35}$$

Thus, the parameter λ may be estimated by the experimental precision of low energy Coulomb scattering. Clearly, as λ becomes very large, corresponding to a wide distribution of events, the potential $\tilde{a}_\varphi^0$ approaches the standard Coulomb potential. In the low energy limit, the equations of motion simplify to

$$M\ddot{x}^k = -\frac{e^2}{4\pi R^2}\hat{x}^k\ e^{-R/\lambda c}\ \Big[1 - \frac{R}{\lambda c}\Big] \qquad M\,\ddot{t} = o(v/c)\ . \tag{36}$$

Numerical solutions for the full equations of motion [5] indicate that the trajectories of the test event are indistinguishable from the Maxwell case, when $\lambda > 10^{-6}$ seconds, corresponding to a photon mass $m_\gamma \sim 10^{-9}$ eV. If we take the accepted experimental error in the photon mass as the actual mass of the photon, then $m_\gamma \simeq 6 \times 10^{-16}$ eV [9], which corresponds to $\lambda \simeq 1$ second.

4 Summary and Conclusions

In their initial formulation of covariant quantum mechanics, Horwitz and Piron [10] constructed an interacting theory of events, mediated by standard τ-independent Maxwell gauge fields. This construction was later deemed incomplete [1, 11] because of the self-consistency problem: on the one hand, particle worldlines are traced out by the evolution of trajectories $x^\mu(\tau)$ under the local influence of the Maxwell field $A^\mu(x)$; on the other hand, $A^\mu(x)$ depends upon the entire particle worldline — it is induced by the Maxwell current $J^\mu(x)$ which can only be found by concatenating the instantaneous current $j^\mu(x, \tau)$, after the worldline has been traced-out. The pre-Maxwell theory, developed to insure a well-posed theory of interactions, contains fields $a^\alpha(x, \tau)$ and a time scale λ required so that λa^μ and $A^\mu = \int d\tau \ a^\mu$ will have the same units. The model of the particle as a distribution of events along the worldline eliminates this problem, and permits comparison with the standard Maxwell theory at low energy. When the range of the photon mass spectrum is taken small enough, then the equations of motion for the off-shell electromagnetic theory coincide with those of the standard Maxwell theory, within experimental limits. The model thus, provides an initial estimate of λ — if λ is larger than about 10^{-6}, then off-shell phenomena must be observable in scattering. Shnerb and Horwitz [12] have given an interpretation of $\hbar/\lambda c^2$ as the width of the mass distribution of the off-shell photons, yielding λc as a coherence length for the photon-matter interaction. The model offered above may be seen as extending this argument to the classical level, as well as providing a mechanism for the narrow width of the photon mass spectrum, based on the structure of matter currents. As $\lambda \to \infty$, the photon mass vanishes, the particle current spreads out along the entire worldline, and the low energy equations of motion become identical with the standard Maxwell case. For λ large but finite, the pre-Maxwell theory remains well-posed, and the expected off-shell phenomena may be compared with the experimental limit. Such a situation suggests that evidence for the pre-Maxwell theory may be found in small deviations from standard electrodynamics.

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