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Selection Rules for Dipole Radiation from a Relativistic Bound State

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Abstract: Recently, in the framework of a relativistic quantum theory with invariant evolution parameter, solutions have been found for the two-body bound state, whose mass spectrum agrees with the nonrelativistic Schrödinger energy spectrum. In this paper, we study the radiative transitions of these states in the dipole approximation and find that the selection rules are identical with those of the usual nonrelativistic theory, expressed in a manifestly covariant form. In addition to the transverse and longitudinal polarizations of the nonrelativistic theory, we find a "scalar" transition, induced by the relative time coordinate, which is of the same type as the longitudinal transition, expressing the Lorentz covariance of the theory.

1. INTRODUCTION

1.1 Covariant quantum mechanics for the two-body system.

The two body relativistic bound state problem has been studied recently[1] in the framework of a relativistic quantum theory with invariant evolution parameter[2]. In this method, the generator of evolution is written in the invariant form

$$K = \frac{p_{1\mu} p_1^\mu}{2M_1} + \frac{p_{2\mu} p_2^\mu}{2M_2} + V(\rho), \tag{1.1}$$

where $\rho = \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 - (t_1 - t_2)^2}$, the invariant spacelike interval between particles. It was shown in ref. 1 that one obtains the nonrelativistic Schrödinger spectrum for potentials

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$V(\rho)$ which have the form of the corresponding nonrelativistic potentials as functions of $\mathbf{r}$ (the correspondence is established by the fact that $t_1 \to t_2$ in the nonrelativistic limit). These spectra are obtained when one restricts the support for the eigenfunctions in spacetime to the subspace of the Minkowski measure space corresponding to the condition $x_1^2 + x_2^2 - t^2 \geq 0$, where we denote by x^μ the *relative* coordinates for the two body system (since the Hamiltonian is quadratic in the four momenta, one separates variables of the center of mass motion and relative motion in the same way as for the nonrelativistic theory). Under this separation, the Hamiltonian takes the form

$$K = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(\rho), \tag{1.2}$$

where

$$P^\mu = p_1^\mu + p_2^\mu \qquad M = M_1 + M_2$$
$$p^\mu = (M_2 p_1^\mu - M_1 p_2^\mu)/M \qquad m = M_1 M_2/M.$$

We shall call the last two terms of (1.2) K_{rel}, the operator generating the relative motion.

The restricted space, called the RMS (Restricted Minkowski Space), is transitive and invariant under the O(2,1) subgroup of O(3,1) leaving invariant the z-axis, dilation in ρ, and translations in z. This subspace is described by the coordinatization

$$x^0 = \rho\ \sinh\beta\ \sin\theta \qquad x^1 = \rho\ \cosh\beta\ \sin\theta\ \cos\phi$$
$$x^2 = \rho\ \cosh\beta\ \sin\theta\ \sin\phi \qquad x^3 = \rho\ \cos\theta \tag{1.3}$$

The spectrum of K_{rel} is determined by the "radial" equation for the ρ dependence of the solution after separation of the angular and hyperbolic variables. In the separation of variables, the Casimir operator for the underlying O(3,1) algebra generated by

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu,$$

i.e.,

$$\Lambda = \frac{1}{2} M^{\mu\nu} M_{\mu\nu} \to \ell(\ell+1) - \frac{3}{4}, \tag{1.4}$$

labels the Legendre functions of $\cos\theta$. The eigenvalues then depend on this ℓ in the same way as for the nonrelativistic bound state problem.

In particular, the solutions for the problem corresponding to the Coulomb potential, yield a spectrum for K_{rel} which coincides with the nonrelativistic Schrödinger spectrum. The observed energies for such a systems are the values of $P^\mu P_\mu$, i.e., $-E^2$ in the center of momentum frame; from (1.2) one obtains, in an expansion in orders of $1/c^2$, the nonrelativistic spectrum with relativistic corrections (in this calculation the total K is given the approximate asymptotic value $-M/2$).

1.2 The induced representation of O(3,1)

It was shown in ref. 1 that the wavefunctions which yield the Schrödinger spectrum form irreducible representations of SU(1,1) — in the double covering of O(2,1) — parameterized by the spacelike vector n_μ stabilized by this O(2,1) (taken in ref. 1 as the z-axis). In ref. 3, an induced representation of SL(2,C) was constructed, by studying the action of the Lorentz group on these wavefunctions, through the transformation of the RMS coordinates x^μ and the frame orientation n_μ. Wavefunctions with support in RMS(n_μ) may be written as functions of n_μ and the coordinates of a standard frame $y \in$ RMS($\mathring{n}_\mu$) since, given the Lorentz transformation $\mathcal{L}$ such that $\mathring{n} = \mathcal{L}(n)\ n$, it follows that $y \in$ RMS($\mathring{n}_\mu$) when $x \in$ RMS(n_μ) and $y = \mathcal{L}(n)\ x$. By choosing $\mathring{n} = (0,0,0,1)$, the parameterization (1.3) may be used for y^μ. Under a Lorentz transformation Λ, the wavefunctions transform as

$$\psi_n(y) \to \psi_n^\Lambda(y) = \psi_{\Lambda^{-1}n}(D^{-1}(\Lambda,n)\ y) \tag{1.5}$$

where Λ acts directly on n_μ and so moves the representations on an orbit covered by this spacelike vector, and acts on y^μ through the O(2,1) little group:

$$D^{-1}(\Lambda,n)\ \mathring{n} = \mathcal{L}(\Lambda^{-1}n)\Lambda\mathcal{L}^T(n)\ \mathring{n} \equiv \mathring{n} \tag{1.6}$$

The matrix $\mathcal{L}^T(n)$ was fixed as a boost in the three-direction, a rotation about the two-axis, followed by a rotation about the one-axis. Thus,

$$\mathcal{L}^T(n) = e^{\gamma\mathcal{M}^{23}} e^{\omega\mathcal{M}^{31}} e^{\alpha\mathcal{M}^{03}} \tag{1.7a}$$

where $(\mathcal{M}^{\alpha\beta})^{\mu\nu} = g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\nu}g^{\beta\mu}$, and so

$$\mathcal{L}^T(n) = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ -\sin\omega\ \sinh\alpha & \cos\omega & 0 & -\sin\omega\ \cosh\alpha \\ \sin\gamma\ \cos\omega\ \sinh\alpha & \sin\gamma\ \sin\omega & \cos\gamma & \sin\gamma\ \cos\omega\ \cosh\alpha \\ \cos\gamma\ \cos\omega\ \sinh\alpha & \cos\gamma\ \sin\omega & -\sin\gamma & \cos\gamma\ \cos\omega\ \cosh\alpha \end{pmatrix} \tag{1.7b}$$

which provides the parameterization of n_μ as

$$n_\mu = \begin{pmatrix} \sinh\alpha \\ -\sin\omega\ \cosh\alpha \\ \sin\gamma\ \cos\omega\ \cosh\alpha \\ \cos\gamma\ \cos\omega\ \cosh\alpha \end{pmatrix} \tag{1.8}$$

By examining the generators $h_{\alpha\beta}(n)$ of (1.5), which form a representation of the O(3,1) Lie algebra, the Casimir operators

$$\hat{c}_1 = \frac{1}{2}h_{\alpha\beta}(n)h^{\alpha\beta}(n) \qquad \hat{c}_2 = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}h_{\alpha\beta}(n)h_{\gamma\delta}(n) \tag{1.9}$$

as well as the operators of the SU(2) subgroup

$$\underline{L}(n)^2 = \frac{1}{2}h_{ij}(n)h^{ij}(n) \qquad L_1(n) = h^{23}(n) = -i\frac{\partial}{\partial\gamma} \tag{1.10}$$

can be constructed as a commuting set. Moreover, the operator Λ (1.4) and the O(2,1) Casimir N^2 commute with this set and, wavefunctions were constructed which are eigenfunctions of the set

$$\{\Lambda, N^2, \hat{c}_1, \hat{c}_2, \underline{L}(n)^2, L_1(n)\}$$

with eigenvalues $Q = \{\ell(\ell+1) - \frac{3}{4}, n^2 - \frac{1}{4}, c_1, c_2, L(L+1), q\}$. The requirement that these wavefunctions form a unitary canonical representation of SL(2,C) (they are in the principal series), imposes the condition $c_1 = \hat{n}^2 - 1 - c_2^2/\hat{n}^2$, where $\hat{n} = n + 1/2$. The wavefunctions in the induced representation have the explicit form

$$\psi_n^Q(y) = R_{n_a \ell}(\rho)\ \Theta_\ell^n(\theta)\ \xi^Q(n_\mu, \beta, \phi) \tag{1.11}$$

where

$$\xi^Q(n_\mu, \beta, \phi) = \sum_{k=0}^{L-\hat{n}} \mathcal{D}_k^Q(\alpha, \omega, \gamma)\ \chi_{n+k}^{-n}(\beta, \phi) \tag{1.12}$$

$$\Theta_\ell^n(\theta) = (1 - \xi^2)^{-\frac{1}{4}} \sqrt{\frac{2\ell+1}{2} \frac{(\ell-n)!}{(\ell+n)!}}\ P_\ell^n(\xi) \tag{1.13}$$

$$\chi_{n+k}^{-n}(\beta, \phi) = B_{n+k,n}(\beta)\ \Phi_{n+k}(\phi) \tag{1.14}$$

$$B_{n+k,n}(\beta) = (1 - \zeta^2)^{\frac{1}{4}} \sqrt{n \frac{(2n+k)!}{k!}}\ P_{n+k}^{-n}(\zeta) \tag{1.15}$$

$$\Phi_{n+k}(\phi) = \frac{1}{\sqrt{2\pi}} e^{i(n+k+\frac{1}{2})\phi} \tag{1.16}$$

$$\mathcal{D}_k^Q(\alpha, \omega, \gamma) = \Xi_{Lk}^{nc_2}(u)\ P_{q,-M_k}^L(z) e^{-iq\gamma} \tag{1.17}$$

$$\Xi_{Lk}^{nc_2}(u) = (-1)^k \sqrt{\frac{(2\hat{n}+k-1)!}{(2\hat{n}-1)!k!}} N_L^Q (1-u^2)^{-\frac{\hat{n}-1}{2}}\ P^L_{-\frac{ic_2}{\hat{n}}, \hat{n}+k}(u) \tag{1.18}$$

where $u = \tanh\alpha$, $z = \sin\omega$, $\xi = \cos\theta$, $\zeta = \tanh\beta$, $M_k = \hat{n} + k$ and N_L^Q is a normalization constant.

These wavefunctions are orthogonal with respect to the measure $d^4y\ d^4n\ \delta(1-n^2)$, where

$$\begin{aligned} \int d^4y &= \int_0^\infty d\rho\ \rho^3 \int_{-\infty}^\infty d\beta\ \cosh\beta \int_0^\pi d\theta\ \sin^2\theta \int_0^{2\pi} d\phi \\ &= \int_0^\infty d\rho\ \rho^3 \int_{-1}^1 d\xi\ \sqrt{1-\xi^2} \int_{-1}^1 d\zeta\ (1-\zeta^2)^{-\frac{3}{2}} \int_0^{2\pi} d\phi \end{aligned} \tag{1.19}$$

$$
\begin{aligned}
\int d^4n\ \delta(1-n^2) &= \frac{1}{2}\int_{-\infty}^{\infty} d\alpha \cosh^2\alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\omega \cos\omega \int_0^{2\pi} d\gamma \\
&= \frac{1}{2}\int_{-1}^{1} \frac{du}{(1-u^2)^2}\int_{-1}^{1} dz \int_0^{2\pi} d\gamma
\end{aligned}
\tag{1.20}
$$

1.3 The electromagnetic interaction and dipole transitions.

Under the minimal substitution $p_\mu \to p_\mu - eA_\mu(x)$, the Hamiltonian (1.1) may be interpreted as describing the electromagnetic interaction of a two body system. This form has been used(4) to describe radiative transitions of the bound state, where it was shown that to lowest order, the transition amplitude in the center of mass frame, is proportional to the matrix element of the relative coordinate x^μ, (which generalizes the dipole transition operator $\mathbf{r}$ in the nonrelativistic case). In this paper we study selection rules and obtain relations for the line widths for electromagnetic transitions of the bound states (1.11). The resulting selection rules coincide with those for the corresponding nonrelativistic case, with $\Delta\ell = \pm 1$ in all cases, $\Delta q = \pm 1$ for the transverse operator $x^\pm$, and $\Delta q = 0$ for the longitudinal and "scalar" operators x^1 and x^0.

2. MATRIX ELEMENTS

2.1 The Transition Operators

We shall compute the matrix elements of the transition operator x^μ between the wavefunctions (1.15) which transform under the induced representation of O(3,1). Since $L_1(n)$ is diagonal in this representation, we will calculate separately the matrix elements for x^1, $x^\pm = x^2 \pm ix^3$, and x^0. The relationship between the x^μ and the y^μ is found from (1.3) and (1.7), which we may use to write

$$
x^0 = \rho\,[\cosh\alpha\ \sinh\beta\ \sin\theta + \sinh\alpha\ \cos\theta] \tag{2.1}
$$

$$
\begin{aligned}
x^1 = \rho\,[&-\sinh\alpha\ \sin\omega\ \sinh\beta\ \sin\theta + \cos\omega\ \cosh\beta\ \sin\theta\ \cos\phi \\
&- \cosh\alpha\ \sin\omega \cos\theta]
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
x^2 = \rho\,[&\sinh\alpha\ \cos\omega\ \sin\gamma\ \sinh\beta\ \sin\theta + \sin\omega\ \sin\gamma\ \cosh\beta\ \sin\theta\ \cos\phi \\
&+ \cos\gamma\ \cosh\beta\ \sin\theta\ \sin\phi + \cosh\alpha\ \cos\omega\ \sin\gamma\ \cos\theta]
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
x^3 = \rho\,[&\sinh\alpha\ \cos\omega\ \cos\gamma\ \sinh\beta\ \sin\theta + \sin\omega\ \cos\gamma\ \cosh\beta\ \sin\theta\ \cos\phi \\
&- \sin\gamma\ \cosh\beta\ \sin\theta\ \sin\phi + \cosh\alpha\ \cos\omega\ \cos\gamma\ \cos\theta]
\end{aligned}
\tag{2.4}
$$

Combining the last two,

$$x^{\pm} = \rho\,[\pm i \sinh\alpha\ \cos\omega\ \sinh\beta\ \sin\theta\ e^{\mp i\gamma} \pm i \sin\omega\ \cosh\beta\ \sin\theta\ \cos\phi\ e^{\mp i\gamma} + \cosh\beta\ \sin\theta\ \sin\phi\ e^{\mp i\gamma} \pm i \cosh\alpha\ \cos\omega\ \cos\theta\ e^{\mp i\gamma}] \tag{2.5}$$

In terms of the parameters u, z, ξ, ζ, these have the form

$$x^0 = \rho\,[\frac{\zeta\sqrt{1-\xi^2}}{\sqrt{1-\zeta^2}} + u\,\xi]\frac{1}{\sqrt{1-u^2}} \tag{2.6}$$

$$x^1 = \rho\,[\frac{\sqrt{1-z^2}\ \sqrt{1-\xi^2}\ \cos\phi}{\sqrt{1-\zeta^2}} - \frac{z\,\xi}{\sqrt{1-u^2}} - \frac{u\ z\ \zeta\ \sqrt{1-\xi^2}}{\sqrt{1-\zeta^2}\ \sqrt{1-u^2}} \tag{2.7}$$

$$x^2 = \rho\,[\frac{u\,\zeta\,\sqrt{1-z^2}\ \sqrt{1-\xi^2}\ \sin\gamma}{\sqrt{1-u^2}\ \sqrt{1-\zeta^2}} + \frac{z\ \sqrt{1-\xi^2}\ \sin\gamma\ \cos\phi}{\sqrt{1-\zeta^2}} + \frac{\sqrt{1-\xi^2}}{\sqrt{1-\zeta^2}}\ \cos\gamma\ \sin\phi + \frac{\xi\ \sqrt{1-z^2}}{\sqrt{1-u^2}}\ \sin\gamma] \tag{2.8}$$

$$x^3 = \rho\,[\frac{u\,\zeta\,\sqrt{1-z^2}\ \sqrt{1-\xi^2}\ \cos\gamma}{\sqrt{1-u^2}\ \sqrt{1-\zeta^2}} + \frac{z\ \sqrt{1-\xi^2}\ \cos\gamma\ \cos\phi}{\sqrt{1-\zeta^2}} - \frac{\sqrt{1-\xi^2}}{\sqrt{1-\zeta^2}}\ \sin\gamma\ \sin\phi + \frac{\xi\ \sqrt{1-z^2}}{\sqrt{1-u^2}}\ \cos\gamma] \tag{2.9}$$

$$x^{\pm} = \rho\,[\pm i\ \frac{u\,\zeta\,\sqrt{1-z^2}\ \sqrt{1-\xi^2}\ e^{\mp i\gamma}}{\sqrt{1-u^2}\ \sqrt{1-\zeta^2}} \pm i\ \frac{z\ \sqrt{1-\xi^2}\ e^{\mp i\gamma}\ \cos\phi}{\sqrt{1-\zeta^2}} + \frac{\sqrt{1-\xi^2}}{\sqrt{1-\zeta^2}}\ e^{\mp i\gamma}\ \sin\phi \pm i\ \frac{\xi\ \sqrt{1-z^2}}{\sqrt{1-u^2}}\ e^{\mp i\gamma}] \tag{2.10}$$

2.2 The x^1 Matrix Element

Using (2.7) for x^1, we will write the transition matrix element, noticing that ρ is a common factor and that there is no γ dependence, as

$$< n_{a'}\ell' n' L' q' c_2' | x^1 | n_a \ell n L q c_2 > = < n_{a'}\ell' |\rho| n_a \ell > < q' | q > \{I_1 - I_2 - I_3\} \tag{2.11}$$

where

$$< n_{a'}\ell' |\rho| n_a \ell > = \int_0^\infty d\rho\ \rho^4\ R^*_{n_{a'}\ell'}(\rho)\ R_{n_a\ell}(\rho) \tag{2.12}$$

$$< q'|q >= \frac{1}{2\pi} \int_0^{2\pi} d\gamma \; e^{iq'\gamma} \; e^{-iq\gamma} = \delta_{qq'} \tag{2.13}$$

$$I_1 =< \ell' n'|\sqrt{1-\xi^2}|\ell n > \sum_{kk'} \; < L'n'c_2'k'|Lnc_2k >< L'qM_{k'}|\sqrt{1-z^2}|LqM_k > \; \times$$

$$\times \; < n'+k' \;\; n'|\frac{1}{\sqrt{1-\zeta^2}}|n+k \;\; n >< M_{k'}|\cos\phi|M_k > \tag{2.14}$$

$$I_2 =< \ell' n'|\xi|\ell n > \sum_{kk'} \; < L'n'c_2'k'|\frac{1}{\sqrt{1-u^2}}|Lnc_2k >< L'qM_{k'}|z|LqM_k > \; \times$$

$$\times \; < n'+k' \;\; n'|n+k \;\; n >< M_{k'}|M_k > \tag{2.15}$$

$$I_3 =< \ell' n'|\sqrt{1-\xi^2}|\ell n > \sum_{kk'} \; < L'n'c_2'k'|\frac{u}{\sqrt{1-u^2}}|Lnc_2k >< L'qM_{k'}|z|LqM_k > \; \times$$

$$\times \; < n'+k' \;\; n'|\frac{\zeta}{\sqrt{1-\zeta^2}}|n+k \;\; n >< M_{k'}|M_k > \tag{2.16}$$

and the matrix elements, for an operator $\mathcal{O}$, are defined as

$$< \ell' n'|\mathcal{O}|\ell n >= \int_{-1}^{1} d\xi \; \sqrt{1-\xi^2} \; \Theta_{\ell'}^{n'*}(\theta) \; \mathcal{O} \; \Theta_\ell^n(\theta) \tag{2.17}$$

$$< n'+k' \;\; n'|\mathcal{O}|n+k \;\; n >= \int_{-1}^{1} d\zeta \; (1-\zeta^2)^{-\frac{3}{2}} B^*_{n'+k',n'}(\beta) \; \mathcal{O} \; B_{n+k,n}(\beta) \tag{2.18}$$

$$< L'q'M_{k'}|\mathcal{O}|LqM_k >= \int_{-1}^{1} dz \; P^{L'*}_{q',-M_k'}(z) \; \mathcal{O} \; P^L_{q,-M_k}(z) \tag{2.19}$$

$$< L'n'c_2'k'|\mathcal{O}|Lnc_2k >= \frac{1}{2} \int_{-1}^{1} \frac{du}{(1-u^2)^2} \Xi^{n'c_2'*}_{L'k'}(u) \; \mathcal{O} \; \Xi^{nc_2}_{Lk}(u) \tag{2.20}$$

To proceed with the evaluation of these integrals, we first notice that the wavefunctions $\Xi^{nc_2}_{Lk}(u)$ are eigenfunctions of the self-adjoint operators[3] $\underline{L}(n)^2$ and N^2, both of which commute with u. Thus, for any $\mathcal{O}(u)$,

$$< L'n'c_2'k'|\mathcal{O}(u)|Lnc_2k >=< Lnc_2'k'|\mathcal{O}(u)|Lnc_2k > \; \delta_{L'L}\delta_{n'n} \tag{2.21}$$

Since, both I_1 and I_3 contain the factor

$$< \ell' n'|\sqrt{1-\xi^2}|\ell n >= \int_{-1}^{1} d\xi \; \sqrt{1-\xi^2} \; \Theta_{\ell'}^{n'*}(\theta) \; \sqrt{1-\xi^2} \; \Theta_\ell^n(\theta) \tag{2.22}$$

$$
\begin{aligned}
&= \sqrt{\frac{2\ell'+1}{2}\frac{(\ell-n')!}{(\ell'+n')!}}\sqrt{\frac{2\ell+1}{2}\frac{(\ell-n)!}{(\ell+n)!}} \times \\
&\qquad \times \int_{-1}^{1} d\xi \, \sqrt{1-\xi^2}\, \{(1-\xi^2)^{-\frac{1}{4}}\, P_{\ell'}^{n'}(\xi)\}\, \sqrt{1-\xi^2}\, \{(1-\xi^2)^{-\frac{1}{4}}\, P_{\ell}^{n}(\xi)\} \\
&= \sqrt{\frac{2\ell'+1}{2}\frac{(\ell-n')!}{(\ell'+n')!}}\sqrt{\frac{2\ell+1}{2}\frac{(\ell-n)!}{(\ell+n)!}} \times \\
&\quad \times \, \{ \, [\frac{2(\ell+n)!}{(2\ell+1)(2\ell-1)(\ell-n-2)!}\delta_{\ell' \, \ell-1} - \frac{2(\ell+n+2)!}{(2\ell+1)(2\ell+3)(\ell-n)!}\delta_{\ell' \, \ell+1}] \, \delta_{n' \, n+1} \\
&\qquad + [\frac{2(\ell+n)!}{(2\ell+1)(2\ell+3)(\ell-n)!}\delta_{\ell' \, \ell+1} - \frac{2(\ell+n)!}{(2\ell+1)(2\ell-1)(\ell-n)!}\delta_{\ell' \, \ell-1}] \, \delta_{n' \, n-1} \}
\end{aligned}
\tag{2.23}
$$

these terms must vanish, and we are left with I_2, which contains

$$
\begin{aligned}
< \ell' n' | \xi | \ell n > &= \int_{-1}^{1} d\xi \, \sqrt{1-\xi^2}\, \Theta_{\ell'}^{n' *}(\theta)\, \xi \, \Theta_{\ell}^{n}(\theta) \\
&= \sqrt{\frac{2\ell'+1}{2}\frac{(\ell-n')!}{(\ell'+n')!}}\sqrt{\frac{2\ell+1}{2}\frac{(\ell-n)!}{(\ell+n)!}} \times \\
&\qquad \times \int_{-1}^{1} d\xi \, \sqrt{1-\xi^2}\, \{(1-\xi^2)^{-\frac{1}{4}}\, P_{\ell'}^{n'}(\xi)\}\, \xi \, \{(1-\xi^2)^{-\frac{1}{4}}\, P_{\ell}^{n}(\xi)\} \\
&= \sqrt{\frac{2\ell'+1}{2}\frac{(\ell-n')!}{(\ell'+n')!}}\sqrt{\frac{2\ell+1}{2}\frac{(\ell-n)!}{(\ell+n)!}} \times \\
&\qquad \times \, \{(\ell-n+1)\sqrt{\frac{(\ell-n)!(\ell+n+1)!}{(\ell+n)!(\ell-n+1)!(2\ell+1)(2\ell+3)}}\; \delta_{\ell' \, \ell+1} \\
&\qquad + (\ell+n)\sqrt{\frac{(\ell-n)!(\ell+n-1)!}{(\ell+n)!(\ell-n-1)!(2\ell+1)(2\ell-1)}}\; \delta_{\ell' \, \ell-1} \} \, \delta_{n' \, n} \\
&\equiv \sum_{i=\pm 1} E_{\ell n}^{(i)} \, \delta_{\ell' \, \ell+i} \, \delta_{n' \, n}
\end{aligned}
\tag{2.24}
$$

where we have used the recursion relations for the Legendre functions[5],

$$
x \, P_{\ell}^{n}(x) = \frac{1}{2\ell+1}[(\ell-n+1)\, P_{\ell+1}^{n}(x) + (\ell+n)\, P_{\ell-1}^{n}(x)]
$$

$$
\sqrt{1-x^2} P_{\ell}^{n}(x) = \frac{1}{2\ell+1}[P_{\ell-1}^{n+1}(x) - P_{\ell+1}^{n+1}(x)]
$$

and

$$\int_{-1}^{1} dx P_\ell^n(x)\ P_{\ell'}^{n'}(x) = \frac{2}{2\ell+1}\frac{(\ell+n)!}{(\ell-n)!}\ \delta_{\ell\ell'}\ \delta_{nn'}$$

The other matrix elements are

$$< n'+k'\ \ n'|n+k\ \ n>< M_{k'}|M_k >= \delta_{n'+k'\ n+k}\ \delta_{M_{k'}\ M_k} = \delta_{nn'}\ \delta_{kk'} \tag{2.25}$$

$$\begin{aligned} < LqM_{k'}|z|LqM_k > &= \int_{-1}^{1} dz\ P_{q,-M_k'}^{L*}(z)\ z\ P_{q,-M_k}^{L}(z) \\ &= -\frac{qM_k}{L(L+1)}\int_{-1}^{1} dz\ P_{q,-M_k'}^{L*}(z)\ P_{q,-M_k}^{L}(z) \\ &= -\frac{qM_k}{L(L+1)}\frac{2}{2L+1}\delta_{M_{k'}\ M_k} \end{aligned} \tag{2.26}$$

where we have used[6]

$$\begin{aligned} x\ P_{ab}^L(x) = &\frac{\sqrt{(L^2-a^2)(L^2-b^2)}}{L(2L+1)}\ P_{ab}^{L-1}(x) + \frac{ab}{L(L+1)}\ P_{ab}^L(x) \\ &+ \frac{\sqrt{((L+1)^2-a^2)((L+1)^2-b^2)}}{(L+1)(2L+1)}\ P_{ab}^{L+1}(x) \end{aligned} \tag{2.27}$$

and

$$\int_{-1}^{1} dx P_{ab}^L(x)\ P_{a'b'}^{L'*}(x) = \frac{2}{2L+1}\delta_{L'L}\delta_{a'a}\delta_{b'b} \tag{2.28}$$

Now,

$$\begin{aligned} < n_{a'}\ell' n' L' q' c_2'|x^1|n_a\ell nLqc_2 >= &\frac{q}{L(L+1)}\frac{2}{2L+1}\ \delta_{qq'}\ \delta_{nn'}\ \delta_{L'L} \\ &\sum_{i=\pm1} E_{\ell n}^{(i)}\delta_{\ell'\ \ell+i}\ < n_{a'}\ell'|\rho|n_a\ell > \sum_{k=0}^{L-\hat{n}} M_k < Lnc_2'k|\frac{1}{\sqrt{1-u^2}}|Lnc_2k > \end{aligned} \tag{2.29}$$

In ref. 3 it was shown that the integral

$$< Lnc_2'k|Lnc_2k >= \frac{1}{2}\int_{-1}^{1}\frac{du}{(1-u^2)^2}\Xi_{L'k'}^{n'c_2'*}(u)\ \Xi_{Lk}^{nc_2}(u)$$

behaves as a distribution proportional to $\delta(c_2 - c_2')$. In the present case,

$$\sum_{k=0}^{L-\hat{n}} M_k < Lnc_2'k|\frac{1}{\sqrt{1-u^2}}|Lnc_2k >= \frac{1}{2}\sum_{k=0}^{L-\hat{n}} M_k\int_{-1}^{1}\frac{du}{(1-u^2)^{\frac{5}{2}}}\Xi_{Lk}^{nc_2'*}(u)\ \Xi_{Lk}^{nc_2}(u) \tag{2.30}$$

the integral is well behaved, except at the limits of integration, $u \to \pm 1$; we may thus consider the integral to be a distribution with the integrand concentrated in the neighborhood of the limits of integration. So,

$$\sum_{k=0}^{L-\hat{n}} M_k < Lnc_2'k|\frac{1}{\sqrt{1-u^2}}|Lnc_2k > \simeq \sum_{k=0}^{L-\hat{n}} M_k \frac{1}{(1-u^2)^{\frac{5}{2}}} \Xi_{L'k'}^{n'c_2'*}(u)\, \Xi_{Lk}^{nc_2}(u)$$
$$= N_L^Q \sum_{k=0}^{L-\hat{n}} M_k \frac{(2\hat{n}+k-1)!}{(2\hat{n}-1)!k!} \lim_{u\to\pm 1} e^{-i[\frac{(c_2-c_2')}{2\hat{n}}+k-\frac{3}{2}]\log(1-u^2)} \tag{2.31}$$

where we have used the fact[3] that

$$\Xi_{Lk}^{nc_2}(u) \sim N_L^Q \sqrt{\frac{(2\hat{n}+k-1)!}{(2\hat{n}-1)!k!}}(1-u^2)^{(k+1+ic_2/\hat{n})/2}$$

in the neighborhood of $u = \pm 1$. Clearly, (2.31) vanishes as a distribution for $k > 1$. But in spaces of smooth functions of $c_2 - c_2'$, we may absorb the $\frac{3}{2}$ into the integration variable as

$$\int dc_2 f(c_2) \sum_{k=0}^{L-\hat{n}} M_k \lim_{u\to\pm 1} e^{-i[\frac{(c_2-c_2')}{2\hat{n}}+k-\frac{3}{2}]\log(1-u^2)}$$
$$= \int dc_2'' f(c_2''+3\hat{n}) \sum_{k=0}^{L-\hat{n}} M_k \lim_{u\to\pm 1} e^{-i[\frac{(c_2''-c_2')}{2\hat{n}}+k]\log(1-u^2)} \tag{2.32}$$

So we may also regard (2.30) as a distribution equivalent to $\delta(c_2 - c_2')$.

We may now examine (2.29) and read off the selection rules for the x^1 induced dipole emission as:

$$\Delta L = \Delta n = \Delta q = \Delta c_2 = 0 \qquad\qquad \Delta\ell = \pm 1 \tag{2.34}$$

2.3 The $x^{\pm}$ Matrix Element

As in the previous section, we must discard all terms in $x^{\pm}$ (2.10) which contain the factor $\sqrt{1-\xi^2}$. The matrix element therefore reduces to

$$< n_{a'}\ell'n'L'q'c_2'|x^1|n_a\ell nLqc_2 > = < n_{a'}\ell'n'L'q'c_2'|\frac{\pm i\,\xi\,\sqrt{1-z^2}}{\sqrt{1-u^2}}\; e^{\mp i\gamma}|n_a\ell nLqc_2 >$$
$$= \pm i < n_{a'}\ell'|\rho|n_a\ell >< \ell'n'|\xi|\ell n > < q'|e^{\mp i\gamma}|q > \;\times$$
$$\times \sum_{kk'} \; < L'q'M_{k'}|\sqrt{1-z^2}|LqM_k >< n'+k' \;\; n'|n+k \;\; n >$$
$$\times \; < M_{k'}|M_k >< L'n'c_2'k'|\frac{1}{\sqrt{1-u^2}}|Lnc_2k > \tag{2.35}$$

From (2.21), and since

$$< q'|e^{\mp i\gamma}|q >= \frac{1}{2\pi}\int_0^{2\pi} d\gamma\; e^{iq'\gamma}\; e^{\mp i\gamma} e^{-iq\gamma} = \delta_{q'\; q\pm 1} \tag{2.36}$$

we need only calculate

$$\begin{aligned} < L\; q\pm 1\; M_{k'}|\sqrt{1-z^2}|L\; q\; M_k >&= \int_{-1}^{1} dz\; P^{L*}_{q\pm 1,-M'_k}(z)\; \sqrt{1-z^2}\; P^L_{q,-M_k}(z) \\ &= \mp\frac{i}{(2L+1)M_k}\sqrt{(L\mp q)(L\pm q+1)} \end{aligned} \tag{2.37}$$

where we have used (2.27), (2.28) and[6]

$$\sqrt{1-z^2}P^L_{ab}(z) = \frac{i}{2b}\{A^-_{\ell a}\; P^L_{a+1\; b}(z) - A^+_{\ell a}\; P^L_{a-1\; b}(z) + z\,[A^-_{\ell b}\; P^L_{a\; b+1}(z) - A^+_{\ell b}\; P^L_{a\; b-1}(z)]\}$$

where $A^{\pm}_{\ell a} = \sqrt{(L\pm a)(L\mp a+1)}$. Combining these results, we find

$$\begin{aligned} < n_{a'}\ell' n' L' q' c'_2|x^{\pm}|n_a \ell n L q c_2 >&= -\frac{\sqrt{(L\mp q)(L\pm q+1)}}{(2L+1)}\delta_{q'\; q\pm 1}\;\delta_{nn'}\;\delta_{L'L}\;\times \\ &\times \sum_{i=\pm 1} E^{(i)}_{\ell n}\delta_{\ell'\;\ell+i}\sum_{k=0}^{L-\hat{n}}\frac{1}{M_k} < Lnc'_2 k|\frac{1}{\sqrt{1-u^2}}|Lnc_2 k > \end{aligned} \tag{2.38}$$

From (2.38), we may read off the selection rules for the dipole emission induced by $x^{\pm}$ as

$$\Delta L = \Delta n = \Delta c_2 = 0 \qquad\qquad \Delta q = \pm 1 \qquad\qquad \Delta\ell = \pm 1 \tag{2.39}$$

2.4 The x^0 Matrix Element

Discarding the term in (2.6) which contains the factor $\sqrt{1-\xi^2}$, we are left with

$$\begin{aligned} < n_{a'}\ell' n' L' q' c'_2|x^0|n_a \ell n L q c_2 >&=< n_{a'}\ell' n' L' q' c'_2|\rho\;\frac{u\;\xi}{\sqrt{1-u^2}}|n_a \ell n L q c_2 > \\ &=< n_{a'}\ell'|\rho|n_a\ell >< q'|q >< \ell' n'|\xi|\ell n >\;\delta_{L'L}\delta_{n'n}\sum_{kk'} < LqM_{k'}|LqM_k >\;\times \\ &\quad\times\; < n'+k'\;\; n'|n+k\;\; n >< M_{k'}|M_k >< Lnc'_2 k'|\frac{u}{\sqrt{1-u^2}}|Lnc_2 k > \\ &=< n_{a'}\ell'|\rho|n_a\ell >\;\delta_{qq'}\;\delta_{nn'}\;\delta_{L'L}\sum_{i=\pm 1} E^{(i)}_{\ell n}\delta_{\ell'\;\ell+i}\sum_{k=0}^{L-\hat{n}} < Lnc'_2 k|\frac{u}{\sqrt{1-u^2}}|Lnc_2 k > \end{aligned} \tag{2.40}$$

The selection rules for the dipole emission induced by x^0 may be read off from this expression as:

$$\Delta L = \Delta n = \Delta q = \Delta c_2 = 0 \qquad\qquad \Delta\ell = \pm 1 \tag{2.41}$$

3. THE SELECTION RULES

The selection rules $\{\Delta\ell = \pm 1; \quad \Delta q = 0, \pm 1\}$ found above coincide with those obtained for dipole radiation in the usual nonrelativistic treatment[7]. In ref. 1, it was shown that the "radial" functions $\frac{1}{\sqrt{\rho}} R_{n_a \ell}(\rho)$ are identical to the corresponding nonrelativistic radial functions (of $|\mathbf{r}|$), and that the energy spectrum depends on n_a and ℓ precisely as in the nonrelativistic case. Therefore, since the ℓ transition must be the usual one, the radiation spectrum obtained from the matrix element $< n_{a'} \ell' | \rho | n_a \ell >$ will also coincide with that given by the nonrelativistic treatment.

As in the nonrelativistic case, ℓ labels the Legendre functions of $\cos\theta$. However, unlike the nonrelativistic case, the wavefunctions do not have a simple angular distribution given by spherical harmonics, and so this point requires some clarification. The space of $x \in \mathrm{RMS}(n_\mu)$ describes the physical space on which the wavefunctions have their support, and the indices on the generators $h_{\mu\nu}(n)$ of the induced representation of O(3,1) refer to the indices of x^μ. But since the wavefunctions are written in terms of the frame orientation n_μ and the parameterization of the standard frame y^μ, the variables θ and ϕ do not directly describe the angular distribution of the wavefunction in the physical space. This distribution is a complicated function of y^μ and n_μ together. Thus, while the transition $\Delta\ell = \pm 1$ is understood in the nonrelativistic case to represent the unit of parity carried off by the photon, the interpretation here is somewhat different.

From (1.3) it is seen that under $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$, the RMS coordinates undergo $(y^0, \mathbf{y}) \to (y^0, -\mathbf{y})$, however (2.1) — (2.4) show that this does not produce $(x^0, \mathbf{x}) \to (x^0, -\mathbf{x})$. Rather, we must define parity through the Lorentz transformation

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1)$$

and use mapping defined in (1.6) to obtain, by straightforward computation

$$D^{-1}(\Lambda_P, n) = \mathcal{L}(\Lambda_P^{-1} n) \Lambda_P \mathcal{L}^T(n) = \mathrm{diag}(1, -1, 1, 1).$$

Thus, parity takes $(y^0, y^1, y^2, y^3) \to (y^0, -y^1, y^2, y^3)$ which is equivalent to $(\theta, \phi) \to (\theta, \pi - \phi)$, and $(n^0, \mathbf{n}) \to (n^0, -\mathbf{n})$ (equivalent to $(\omega, \gamma) \to (-\omega, \gamma + \pi)$). It is easily checked that this combination of transformations indeed produces $(x^0, \mathbf{x}) \to (x^0, -\mathbf{x})$. Under these parity transformations, the wavefunctions

$$\psi_n^Q(y) = R_{n_a \ell}(\rho)\, \Theta_\ell^n(\theta) \sum_{k=0}^{L-\hat{n}} \Xi_{Lk}^{n c_2}(u)\, P_{q,-M_k}^L(z)\, e^{-iq\gamma}\, B_{n+k,n}(\beta)\, \Phi_{n+k}(\phi)$$

become

$$\psi_n^Q(y) = (i)^{2L}\ R_{n_a\ell}(\rho)\ \Theta_\ell^n(\theta)\ \sum_{k=0}^{L-\hat{n}}\ \Xi_{Lk}^{nc_2}(u)\ P_{q,+M_k}^L(z)\ e^{-iq\gamma}\ B_{n+k,n}(\beta)\ \Phi_{n+k}(\phi)^*$$

where we have used[8]

$$P_{a,b}^L(-z) = (i)^{2L-2a-2b}\ P_{a,-b}^L(z)$$

Thus, parity takes the wavefunctions to a set of inequivalent representations which were discussed in ref. 1.

On the other hand, if we carry out a spacetime inversion[9] through $\theta \to \theta + \pi$, which produces $y^\mu \to -y^\mu$, then the wavefunction simply acquires the phase $(-1)^{\ell - n}$, showing that it is an eigenfunction of this operation (and is in a well-defined state of spacetime inversion parity). So, we may understand the transition rule for ℓ as the photon carrying off a unit of spacetime inversion parity, generalizing the space inversion parity which the photon carries off in the nonrelativistic treatment. A more complete discussion of the group theoretical structure of the discrete symmetries in the framework of the induced representation will be given elsewhere.

The interpretation of $\Delta q = 0, \pm 1$ as corresponding to one of the usual selection rules, leads to the identification of q with the magnetic quantum number, which as the eigenvalue of $L_1(n)$ it must be. From the discussion above, it should not be surprising that the magnetic quantum number q is conjugate to the angle γ (one of the parameters of n_μ), since rotations about the one-axis affect x^μ *only* through the rotation of the frame orientation n_μ, and do not act on y^μ. To see this, note that from (2.1), (2.2) and (2.5), the transformation $\gamma \to \gamma - \Delta\gamma$ takes $(x^0, x^1, x^\pm) \to (x^0, x^1, e^{\pm i \Delta\gamma} x^\pm)$, which is the spherical representation for a rotation by $\Delta\gamma$ about the one-axis. This structure can be understood by regarding (1.6) as defining a homomorphism of O(3,1) onto O(2,1). It can be seen from (1.7a) that the rotations about the one-axis are elements of the kernel of this mapping:

$$D^{-1}(e^{\gamma \mathcal{M}^{23}}, n) \equiv 1$$

and leave the y^μ invariant. (It is for this reason that $L_1(n)$ has the simple form $-i\partial/\partial\gamma$ — it annihilates the y^μ coordinates). Thus, the transition rule for q represents a *real* shift in the diagonal component of angular momentum in the x^μ coordinates.

We have seen that the conservation of the eigenvalues L and n in the matrix elements of x^μ implies the vanishing of the matrix element $<\ell' n'| \sin\theta |\ell n>$, leaving only the terms containing $<\ell' n| \cos\theta |\ell n>$ in the calculations. Since this term arises only from the $y^3 = \rho\ \cos\theta$ component of y^μ, the terms of x^μ which contribute to calculations are of the form $\mathcal{L}(n)^{3\mu} y_3$. The 3-column of $\mathcal{L}^T$ is precisely n_μ, so the calculation factors as

$$\begin{aligned}
< n_{a'}\ell' n' L' q' c_2' | x^\mu | n_a \ell n L q c_2 > &=< n_{a'}\ell' n' L' q' c_2' | \rho\ \cos\theta\ n^\mu | n_a \ell n L q c_2 > \\
&=< n_{a'}\ell' |\rho| n_a \ell >< \ell' n | \cos\theta | \ell n >< n' L' q' c_2' | n^\mu | n L q c_2 > \\
&=< n_{a'}\ell' |\rho| n_a \ell >< \ell' n | \cos\theta | \ell n >\ \times \\
&\quad \times\ < n L q' c_2 | n^\mu | n L q c_2 >\ \delta_{nn'}\ \delta_{LL'}\ \delta(c_2 - c_2').
\end{aligned}$$

which shows directly that it is the orientation of n_μ which determines the transition in q.

Finally, we point out that x^1 (the longitudinal polarization) and x^0 (the "scalar" polarization) induce the same $\Delta q = 0$ transition for the relativistic case, which has a natural interpretation in terms of the Gupta-Bleuler quantization of the photon. This relationship shows that the wavefunctions act correctly as representations of the the Lorentz group.

It is our very deep pleasure to dedicate this paper to Fritz Rohrlich on the occasion of his formal retirement. To one of us (LPH) he has been a valued friend and colleague for many years, and to both of us a source of knowledge and inspiration.

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