Discrete Symmetries of Off-Shell Electromagnetism*

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*Stueckelberg-Schrodinger Relativistic Quantum Theory and its Associated 5D Electromagnetic Theory

Stueckelberg's Model of Pair Creation/Annihilation

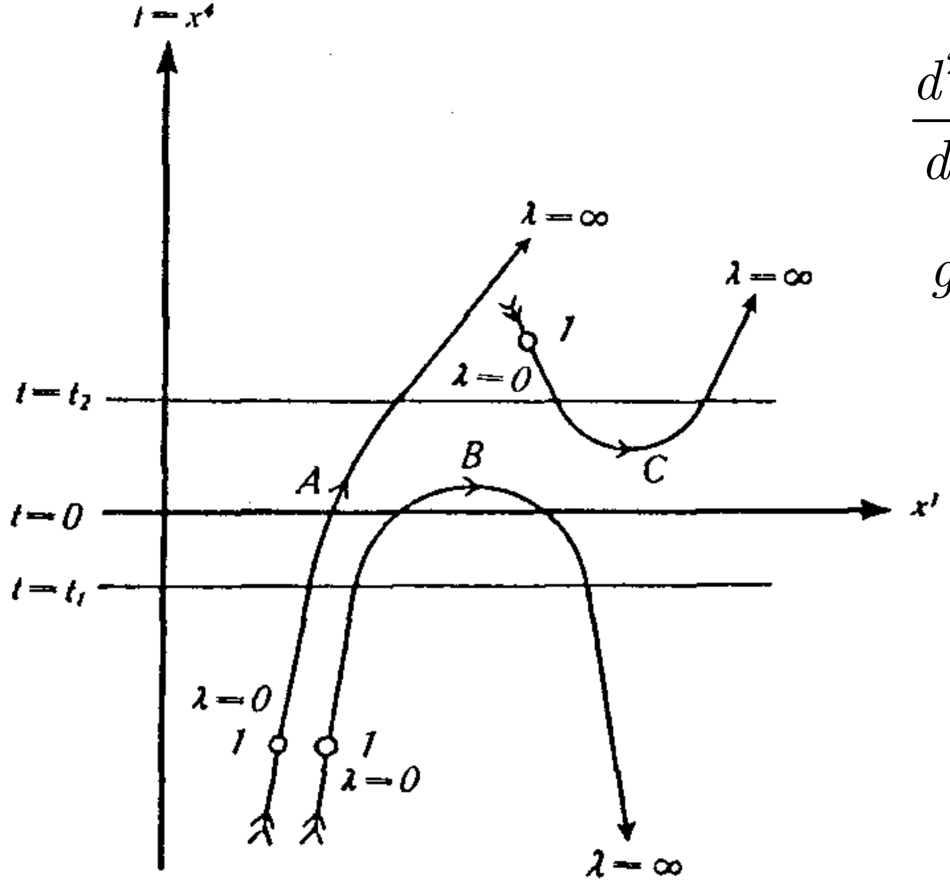


Fig. 1.

Lignes d'univers: *A*. type habituel (à chaque temps $t = x^4$ correspond *un seul* x^1 représentant l'endroit de la particule); *B*. type annihilation (à chaque $t = x^4 \ll 0$ correspondent *deux valeurs de* x^1 représentant les endroits d'une paire de particules qui vont s'annihiler pour $t \sim 0$); *C*. type production de paire (à chaque $t = x^1 \gg 0$ correspondent *deux valeurs de* x^1 etc.).

$$\frac{d^2 q^\mu}{d\lambda^2} = -\Gamma^\mu_{\nu\rho} \frac{dq^\nu}{d\lambda}\frac{dq^\rho}{d\lambda} + eB^{\mu\nu} g_{\nu\rho} \frac{dq^\rho}{d\lambda} + K^\mu$$

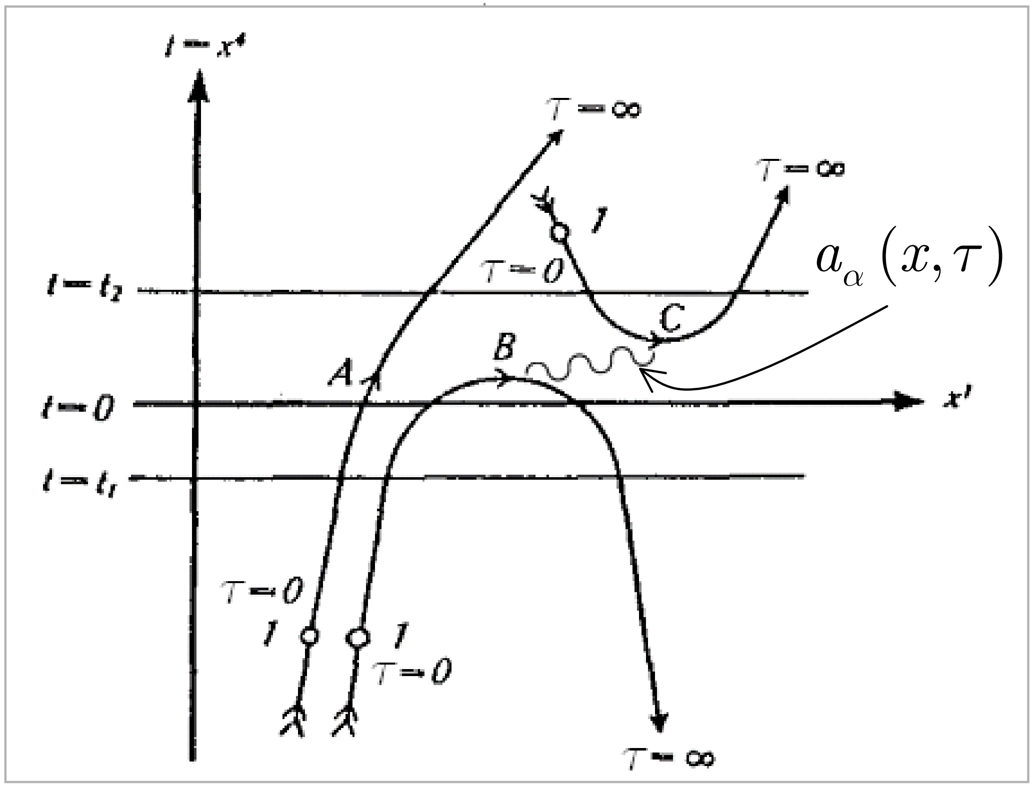
$$g_{\mu\nu} = \mathrm{diag}(-1,1,1,1), \qquad \mu,\nu,\rho = 0,\cdots,3$$

$$m^2 = -g_{\nu\rho} \frac{dq^\nu}{d\lambda}\frac{dq^\rho}{d\lambda}$$

Classical Off-Shell Electrodynamics

Lorentz force
$$M\left[\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right]=e_0 f^{\mu\alpha}\frac{dx_\alpha}{d\tau}=e_0\left[f^{\mu\nu}\frac{dx_\nu}{d\tau}-\sigma f^{\mu 5}\right]$$

Flat metric
$$g_{\alpha\beta}=\mathrm{diag}(-1,1,1,1,\sigma),\quad \mu,\nu,\rho=0,\cdots,3 \quad \alpha,\beta,\gamma=0,\cdots,3,5 \qquad x^5=\tau$$



$$f_{\alpha\beta}(x,\tau)=\partial_\alpha a_\beta(x,\tau)-\partial_\beta a_\alpha(x,\tau)$$

Dynamic mass conservation
$$\frac{d}{d\tau}\left(-\frac{1}{2}M\dot{x}^2\right)=-M\dot{x}^\mu\ddot{x}_\mu$$
$$=-\dot{x}^\mu e_0\left(f_{\mu5}+f_{\mu\nu}\dot{x}^\nu\right)$$
$$=e_0 f_{5\alpha}\dot{x}^\alpha$$

By homogeneous field equations
$$f_{5\mu}=0 \quad\Rightarrow\quad \partial_\tau f^{\mu\nu}=0$$

Off-Shell Electromagnetic Fields

Action for Off-Shell quantum theory

$$\mathbf{S} = \int d^4x d\tau \left\{ \psi^*(i\partial_\tau + e_0 a_5)\psi - \frac{1}{2M}\psi^*(p_\mu - e_0 a_\mu)(p^\mu - e_0 a^\mu)\psi - \frac{\lambda}{4} f_{\alpha\beta} f^{\alpha\beta} \right\}$$

Gauge symmetry

$$\psi(x,\tau) \rightarrow e^{ie_0\Lambda(x,\tau)}\psi(x,\tau)$$

$$a_\alpha(x,\tau) \rightarrow a_\alpha(x,\tau) + \partial_\alpha \Lambda(x,\tau)$$

Field equations

$$\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda} j^\alpha = e j^\alpha \qquad \epsilon^{\alpha\beta\gamma\delta\epsilon}\partial_\alpha f_{\beta\gamma} = 0$$

Field equations in 3-vector form

$$\nabla \cdot \mathbf{e} - \frac{\partial}{\partial \tau}\epsilon^0 = e j^0 \qquad \nabla \times \mathbf{e} + \frac{\partial}{\partial t}\mathbf{h} = 0$$

$$\nabla \times \mathbf{h} - \frac{\partial}{\partial t}\mathbf{e} - \frac{\partial}{\partial \tau}\boldsymbol{\epsilon} = e\mathbf{j} \qquad \nabla \cdot \mathbf{h} = 0$$

$$\nabla \cdot \boldsymbol{\epsilon} + \frac{\partial}{\partial t}\epsilon^0 = e j^5 \qquad \nabla \times \boldsymbol{\epsilon} - \sigma \frac{\partial}{\partial \tau}\mathbf{h} = 0$$

$$\nabla \epsilon^0 + \sigma \frac{\partial}{\partial \tau}\mathbf{e} + \frac{\partial}{\partial t}\boldsymbol{\epsilon} = 0$$

$$e_i = f^{0i} \qquad h_i = \frac{1}{2}\epsilon_{ijk} f^{jk}$$

$$\epsilon^i = f^{5i} \qquad \epsilon^0 = f^{50}$$

Connection With Maxwell Theory

Five dimensional conserved current $$\partial_\alpha j^\alpha = \partial_\mu j^\mu + \partial_\tau j^5 = 0$$

$$j^5 \equiv \rho = \left| \psi(x,\tau) \right|^2 \qquad j^\mu = \frac{-i}{2M}\left\{\psi^*(\partial^\mu - ie_0 a^\mu)\psi - \psi(\partial^\mu + ie_0 a^\mu)\psi^*\right\}$$

Divergenceless Maxwell Current

$$\partial_\mu j^\mu = -\partial_\tau \rho \neq 0$$

$$\rho \xrightarrow[\tau\to\pm\infty]{} 0 \;\Rightarrow\; \partial_\mu J^\mu(x) = \partial_\mu \int_{-\infty}^{\infty} d\tau \; j^\mu(x,\tau) = 0$$

$$f^{5\mu} \xrightarrow[\tau\to\pm\infty]{} 0 \;\Rightarrow\; \left.\begin{array}{r} \partial_\beta f^{\alpha\beta} = e j^\alpha \\ \epsilon^{\alpha\beta\gamma\delta\epsilon}\partial_\alpha f_{\beta\gamma} = 0 \end{array}\right\} \to \left\{\begin{array}{l} \partial_\nu F^{\mu\nu} = e J^\mu \\ \epsilon^{\mu\nu\rho\lambda}\partial_\mu F_{\nu\rho} = 0 \end{array}\right.$$

Concatenation

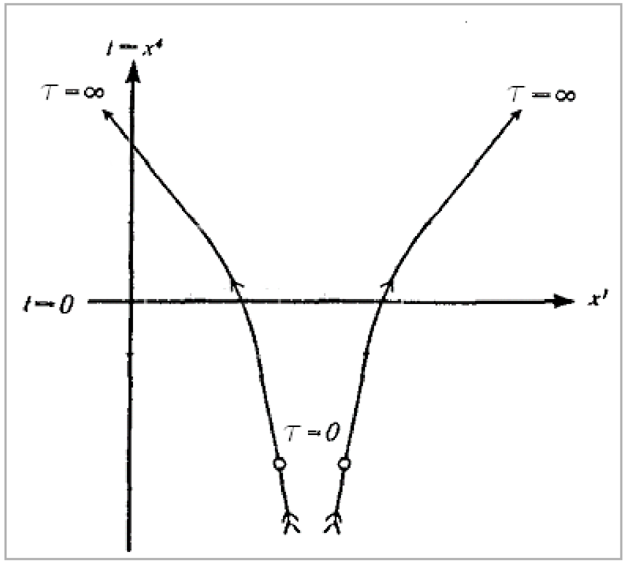
$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau \; f^{\mu\nu}(x,\tau)$$

$$A^\mu(x) = \int_{-\infty}^{\infty} d\tau \; a^\mu(x,\tau)$$

Classical Scattering Problem

Wave equation $$\partial_\alpha \partial^\alpha f^{\beta\gamma} = (\partial_\mu \partial^\mu + \sigma\ \partial_\tau^2) f^{\beta\gamma} = -e(\partial^\beta j^\gamma - \partial^\gamma j^\beta)$$

Green's function $$G(x,\tau) = -\frac{1}{4\pi}\delta(x^2)\delta(\tau) - \frac{1}{2\pi^2}\frac{\partial}{\partial x^2}\ \frac{\theta(-\sigma g_{\alpha\beta}x^\alpha x^\beta)}{\sqrt{-\sigma g_{\alpha\beta}x^\alpha x^\beta}}$$



Classical current $$j^\alpha(x,\tau) = \dot{X}^\alpha(\tau)\ \delta^4(x^\mu - X^\mu(\tau))$$

Induced field of "static" event $$\left(X^0(\tau), \mathbf{X}(\tau)\right) = (\tau, \mathbf{0})$$

$$a^0(x,\tau) = -\frac{e}{4\pi R}\ \frac{1}{2}[\delta(t-R-\tau) + \delta(t+R-\tau)]$$

$$\mathbf{a}(x,\tau) = 0$$

$$a^5(x,\tau) = a^0(x,\tau)$$

Classical Coulomb Problem

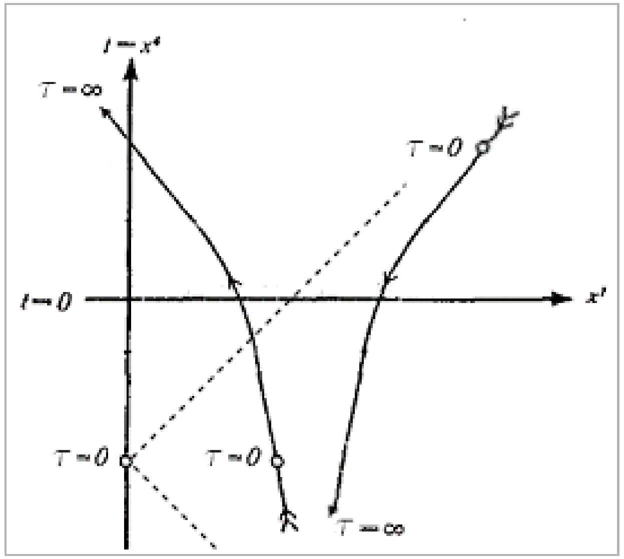
Modified action

$$S_{em-kinetic} = S^0_{em-kinetic} + \frac{\lambda^3}{4}\int d^4x\, d\tau\ \ (\partial_\tau f^{\alpha\beta}(x,\tau))(\partial_\tau f_{\alpha\beta}(x,\tau))$$

Modified Coulomb potential

$$a^0(x,\tau) = -\frac{e}{4\pi R}\ \frac{1}{2\lambda}\ \frac{1}{2}\ \left[e^{-\left|t-R-\tau\right|/\lambda} + e^{-\left|t+R-\tau\right|/\lambda}\right] = \frac{1}{2\lambda}\tilde{a}^0$$

$$\tilde{\boldsymbol{\epsilon}} = \tilde{\mathbf{e}} = -\nabla \tilde{a}^0(x,\tau)$$



Modified Lorentz force

$$M\,\frac{d^2\mathbf{x}}{d\tau^2} = \frac{e}{2}\tilde{\mathbf{e}}\left(\frac{dx^0}{d\tau} - \sigma\right)$$

Low (positive) energy approximation

$$M\,\frac{d^2\mathbf{x}}{d\tau^2} \simeq \frac{e}{2}\tilde{\mathbf{e}}\,(1-\sigma) = \begin{cases} e\tilde{\mathbf{e}} & ,\ \sigma = -1 \\ 0 & ,\ \sigma = +1 \end{cases}$$

Particle-antiparticle interaction

$$M\,\frac{d^2\mathbf{x}}{d\tau^2} = \frac{e}{2}\tilde{\mathbf{e}}\left(\frac{dx^0}{d\tau} + 1\right) \qquad \frac{dx^0}{d\tau} = \frac{E}{m} = -\left(1 + \frac{\text{kinetic energy}}{\text{mass}}\right)$$

Discrete Symmetries of pre-Maxwell Fields

Form invariances: space and time inversion

$$x=\left(x^0,\mathbf{x}\right)\underset{P}{\longrightarrow}x_P=\left(x_p^0,\mathbf{x}_p\right)=\left(x^0,-\mathbf{x}\right)$$

$$x=\left(x^0,\mathbf{x}\right)\underset{T}{\longrightarrow}x_T=\left(x_T^0,\mathbf{x}_T\right)=\left(-x^0,\mathbf{x}\right)$$

Space inversion	Time inversion
$M\frac{d^2x_P^0}{d\tau^2}=e_0\left[\mathbf{e}_P\cdot\frac{d\mathbf{x}_P}{d\tau}-\sigma\epsilon_P^0\right]$	$-M\frac{d^2x^0}{d\tau^2}=e_0\left[\mathbf{e}_T\cdot\frac{d\mathbf{x}}{d\tau}-\sigma\epsilon_T^0\right]$
$M\frac{d^2\mathbf{x}_P}{d\tau^2}=e_0\left[\mathbf{e}_P\frac{dx_P^0}{d\tau}+\frac{d\mathbf{x}_P}{d\tau}\times\mathbf{h}_P-\sigma\boldsymbol{\epsilon}_P\right]$	$M\frac{d^2\mathbf{x}}{d\tau^2}=e_0\left[\mathbf{e}_T\left(-\frac{dx^0}{d\tau}\right)+\frac{d\mathbf{x}}{d\tau}\times\mathbf{h}_T-\sigma\epsilon_T\right]$
$M\frac{d^2x^0}{d\tau^2}=e_0\left[\mathbf{e}_P\cdot\left(-\frac{d\mathbf{x}}{d\tau}\right)-\sigma\epsilon_P^0\right]$	$M\frac{d^2x_T^0}{d\tau^2}=e_0\left[\mathbf{e}_T\cdot\frac{d\mathbf{x}_T}{d\tau}-\sigma\epsilon_T^0\right]$
$-M\frac{d^2\mathbf{x}}{d\tau^2}=e_0\left[\mathbf{e}_P\frac{dx^0}{d\tau}-\frac{d\mathbf{x}}{d\tau}\times\mathbf{h}_P-\sigma\boldsymbol{\epsilon}_P\right]$	$M\frac{d^2\mathbf{x}_T}{d\tau^2}=e_0\left[\mathbf{e}_T\frac{dx_T^0}{d\tau}+\frac{d\mathbf{x}_T}{d\tau}\times\mathbf{h}_T-\sigma\epsilon_T\right]$
$\mathbf{e}_P=-\mathbf{e}$ $\epsilon_P^0=\epsilon^0$	$\mathbf{e}_T=\mathbf{e}$ $\epsilon_T^0=-\epsilon^0$
$\mathbf{h}_P=\mathbf{h}$ $\boldsymbol{\epsilon}_P=-\boldsymbol{\epsilon}$	$\mathbf{h}_T=\mathbf{h}$ $\boldsymbol{\epsilon}_T=\boldsymbol{\epsilon}$

Discrete Symmetries of Currents and Potentials

Field equations

$$\nabla\cdot\mathbf{e}-\frac{\partial}{\partial\tau}\epsilon^0=ej^0 \qquad \nabla\times\mathbf{e}+\frac{\partial}{\partial t}\mathbf{h}=0$$

$$\nabla\times\mathbf{h}-\frac{\partial}{\partial t}\mathbf{e}-\frac{\partial}{\partial\tau}\boldsymbol{\epsilon}=e\mathbf{j} \qquad \nabla\cdot\mathbf{h}=0$$

$$\nabla\cdot\boldsymbol{\epsilon}+\frac{\partial}{\partial t}\epsilon^0=ej^5 \qquad \nabla\times\boldsymbol{\epsilon}-\sigma\frac{\partial}{\partial\tau}\mathbf{h}=0$$

$$\nabla\epsilon^0+\sigma\frac{\partial}{\partial\tau}\mathbf{e}+\frac{\partial}{\partial t}\boldsymbol{\epsilon}=0$$

Space inversion	Time inversion
$j^0_P=j^0 \quad \mathbf{j}_P=-\mathbf{j} \quad j^5_P=j^5$	$j^0_T=-j^0 \quad \mathbf{j}_T=\mathbf{j} \quad j^5_T=j^5$
$e^i=\partial^0a^i-\partial^ia^0 \qquad \mathbf{h}^i=\frac{1}{2}\varepsilon^{ijk}\partial_j a_k$	$\epsilon^i=\partial^5a^i-\partial^ia^5 \qquad \epsilon^0=\partial^5a^0-\partial^0a^5$
$\left(a^0,\mathbf{a},a^5\right)_P=\left(a^0,-\mathbf{a},a^5\right)$	$\left(a^0,\mathbf{a},a^5\right)_T=\left(-a^0,\mathbf{a},a^5\right)$

Spacetime Symmetries in Quantum Mechanics

Stueckelberg-Schrodinger equation

$$(i\partial_\tau + e_0 a_5)\psi(x,\tau) = -\frac{1}{2M}(\partial^\mu - ie_0 a^\mu)(\partial_\mu - ie_0 a_\mu)\psi(x,\tau)$$

Space inversion

$$(\partial^0 - ie_0 a^0)_P = (\partial^0 - ie_0 a^0)$$

$$(\partial^i - ie_0 a^i)_P = -(\partial^i - ie_0 a^i)$$

$$(i\partial_\tau + e_0 a_5)_P = (i\partial_\tau + e_0 a_5)$$

$$\psi_P(x_P,\tau) = \psi(x,\tau)$$

$$\psi_P(x,\tau) = \psi(x^0, -\mathbf{x}, \tau)$$

Time inversion

$$(\partial^0 - ie_0 a^0)_T = -(\partial^0 - ie_0 a^0)$$

$$(\partial^i - ie_0 a^i)_T = (\partial^i - ie_0 a^i)$$

$$(i\partial_\tau + e_0 a_5)_T = (i\partial_\tau + e_0 a_5)$$

$$\psi_T(x_T,\tau) = \psi(x,\tau)$$

$$\psi_T(x,\tau) = \psi(-x^0, \mathbf{x}, \tau)$$

Charge Symmetry in Quantum Mechanics

Stueckelberg-Schrodinger equation

$$(i\partial_\tau + e_0 a_5)\psi(x,\tau) = -\frac{1}{2M}(\partial^\mu - ie_0 a^\mu)(\partial_\mu - ie_0 a_\mu)\psi(x,\tau)$$

Charge conjugation

$$(i\partial_\tau - e_0 a_5)\psi_C(x,\tau) = -\frac{1}{2M}(\partial^\mu + ie_0 a^\mu)(\partial_\mu + ie_0 a_\mu)\psi_C(x,\tau)$$

Complex conjugation

$$(-i\partial_\tau + e_0 a_5)\psi^*(x,\tau) = -\frac{1}{2M}(\partial^\mu + ie_0 a^\mu)(\partial_\mu + ie_0 a_\mu)\psi^*(x,\tau)$$

τ - inversion

$$(i\partial_\tau + e_0 a_{5\mathcal{T}})\psi^*(x,-\tau) = -\frac{1}{2M}(\partial^\mu + ie_0 a^\mu_{\mathcal{T}})(\partial_\mu + ie_0 a_{\mu\mathcal{T}})\psi^*(x,-\tau)$$

Charge conjugation operation

$$\psi(x,\tau) \underset{C}{\rightarrow} \psi_C(x,\tau) = \psi^*(x,-\tau)$$

$$a^\mu(x,\tau) \underset{C}{\rightarrow} a^\mu_C(x,\tau) = a^\mu(x,-\tau)$$

$$a^5(x,\tau) \underset{C}{\rightarrow} a^5_C(x,\tau) = -a^5(x,-\tau)$$

Charge Symmetry of Classical Lorentz Force

Charge conjugation of gauge potentials

$$a^\mu(x,\tau) \underset{C}{\longrightarrow} a_C^\mu(x,\tau) = a^\mu(x,-\tau)$$

$$a^5(x,\tau) \underset{C}{\longrightarrow} a_C^5(x,\tau) = -a^5(x,-\tau)$$

Charge conjugation of field strengths

$$e^k = f^{0k} = \partial^0 a^k - \partial^k a^0 \quad \underset{C}{\longrightarrow} \quad e^k$$

$$h^k = \varepsilon^{kij}\partial_i a_j \quad \underset{C}{\longrightarrow} \quad h^k$$

$$\epsilon^k = f^{5k} = \sigma\partial_\tau a^k - \partial^k a_5 \quad \underset{C}{\longrightarrow} \quad -\epsilon^k$$

$$\epsilon^0 = f^{50} = \sigma\partial_\tau a^0 - \partial^0 a_5 \quad \underset{C}{\longrightarrow} \quad -\epsilon^0$$

Charge conjugation of Lorentz force

$$M\frac{d^2 x_{\mathcal{C}}^0}{d\tau^2} = e_0\left[\mathbf{e}_{\mathcal{C}} \cdot \left(\frac{d\mathbf{x}}{d\tau}\right)_{\mathcal{C}} - \sigma\epsilon_{\mathcal{C}}^0\right] \qquad M\frac{d^2 \mathbf{x}_{\mathcal{C}}}{d\tau^2} = e_0\left[\mathbf{e}_{\mathcal{C}}\left(\frac{dx^0}{d\tau}\right)_{\mathcal{C}} + \left(\frac{d\mathbf{x}}{d\tau}\right)_{\mathcal{C}} \times \mathbf{h}_{\mathcal{C}} - \sigma\boldsymbol{\epsilon}_{\mathcal{C}}\right]$$

$$M\frac{d^2 x^0}{d\tau^2} = -e_0\left[\mathbf{e} \cdot \frac{d\mathbf{x}}{d\tau} - \sigma\epsilon^0\right] \qquad M\frac{d^2 \mathbf{x}}{d\tau^2} = -e_0\left[\mathbf{e}\frac{dx^0}{d\tau} + \frac{d\mathbf{x}}{d\tau} \times \mathbf{h} - \sigma\boldsymbol{\epsilon}\right]$$

Charge Symmetry of Field Equations

Charge conjugation of field strengths

$$e^k \underset{C}{\longrightarrow} e^k \qquad h^k \underset{C}{\longrightarrow} h^k$$

$$\epsilon^k \underset{C}{\longrightarrow} -\epsilon^k \qquad \epsilon^0 \underset{C}{\longrightarrow} -\epsilon^0$$

Inhomogeneous equations

$$\nabla \cdot \mathbf{e} - \left(-\partial_\tau\right)\left(-\epsilon^0\right) = e j_C^0$$

$$\nabla \cdot \mathbf{e} - \partial_\tau \epsilon^0 = e j_C^0 \quad \Rightarrow \quad j_C^0 = j^0$$

$$\nabla \times \mathbf{h} - \partial_0 \mathbf{e} - \left(-\partial_\tau\right)(-\epsilon) = e \mathbf{j}_C$$

$$\nabla \times \mathbf{h} - \partial_0 \mathbf{e} - \partial_\tau \epsilon = e \mathbf{j}_C \quad \Rightarrow \quad \mathbf{j}_C = \mathbf{j}$$

$$\nabla \cdot (-\epsilon) + \partial_0 \left(-\epsilon^0\right) = e j_C^5$$

$$\nabla \cdot \epsilon + \partial_0 \epsilon^0 = -e j_C^5 \quad \Rightarrow \quad j_C^5 = -j^5$$

Homogeneous equations

$$\nabla \cdot \mathbf{h} = 0$$

$$\nabla \times \mathbf{e} + \partial_0 \mathbf{h} = 0$$

$$\nabla \times \boldsymbol{\epsilon} - \sigma \partial_\tau \mathbf{h} = 0$$

$$\nabla \epsilon^0 + \sigma \partial_\tau \mathbf{e} + \partial_0 \boldsymbol{\epsilon} = 0$$

Summary

- Under space and time inversions, potentials and currents behave like the corresponding vector components.
- The quantum theory is manifestly form invariant under space and time inversion.
- Spacetime inversions do not affect the parameter τ or the Lorentz-invariant “5-component” of currents and gauge fields.
- Charge conjugation in the quantum theory requires inversion of the parameter τ and the corresponding field component a^5.
- The charge conjugation operation has the same affect in the quantum and classical theory.
- The antiparticle may be recognized as a particle with $E < 0$. The reversal of quantum numbers is observed in the laboratory when the observer uses the laboratory clock the parameter which orders the events. For antiparticles, this re-parameterization is equivalent to a reversal of τ which performs the charge conjugation in this formalism.