Charge Duality in Off-Shell Electromagnetism*

Martin Land
Hadassah College, Jerusalem

*Stueckelberg-Schrodinger Relativistic Quantum Theory and its Associated 5D Electromagnetic Theory

Off-Shell Electromagnetic Fields

Field equations

$$\partial_\beta f^{\alpha\beta} = e j^\alpha \qquad \epsilon^{\alpha\beta\gamma\delta\epsilon} \partial_\alpha f_{\beta\gamma} = 0$$

$$f_{\alpha\beta}(x,\tau) = \partial_\alpha a_\beta(x,\tau) - \partial_\beta a_\alpha(x,\tau)$$

Flat metric

$$g_{\alpha\beta} = \mathrm{diag}(-1,1,1,1,\sigma), \quad \mu,\nu,\rho = 0,\cdots,3$$

$$\alpha,\beta,\gamma = 0,\cdots,3,5 \qquad x^5 = \tau$$

Connection with Maxwell theory

$$\rho \xrightarrow[\tau\to\pm\infty]{} 0 \;\Rightarrow\; \partial_\mu J^\mu(x) = \partial_\mu \int_{-\infty}^{\infty} d\tau \; j^\mu(x,\tau) = 0$$

$$f^{5\mu} \xrightarrow[\tau\to\pm\infty]{} 0 \;\Rightarrow\; \left.\begin{matrix} \partial_\beta f^{\alpha\beta} = e j^\alpha \\ \epsilon^{\alpha\beta\gamma\delta\epsilon}\partial_\alpha f_{\beta\gamma} = 0 \end{matrix}\right\} \to \left\{\begin{matrix} \partial_\nu F^{\mu\nu} = e J^\mu \\ \epsilon^{\mu\nu\rho\lambda}\partial_\mu F_{\nu\rho} = 0 \end{matrix}\right.$$

$$F^{\mu\nu}(x) = \int_{-\infty}^{\infty} d\tau \; f^{\mu\nu}(x,\tau)$$

$$A^\mu(x) = \int_{-\infty}^{\infty} d\tau \; a^\mu(x,\tau)$$

Space-Time (Clifford) Algebra

Vector Product	$$ab = \frac{1}{2}(ab+ba) + \frac{1}{2}(ab-ba) = a\cdot b + a\wedge b$$
Multivector	$$\begin{aligned} A &= A_0 + A_1 + A_2 + A_3 + \cdots + A_D \\ &= A_0 + A_1{}^i \boldsymbol{e}_i + A_2{}^{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j + \cdots + A_D{}^{i_0 i_2 \cdots i_{D-1}} \boldsymbol{e}_{i_0} \wedge \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_{D-1}} \end{aligned}$$
Basis: 2^n dimensions	$$\left\{1, \boldsymbol{e}_i, \boldsymbol{e}_i \wedge \boldsymbol{e}_j, \boldsymbol{e}_i \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_k, \cdots, \boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_{D-1}\right\}$$
Multivector Products	$$\begin{aligned} aA_r &= a\left(a_1 \wedge a_2 \wedge \cdots \wedge a_r\right) = a\cdot A_r + a\wedge A_r \\ a\cdot A_r &= \frac{1}{2}\left[aA_r - (-1)^r A_r a\right] \\ &= \sum_{i=1}^{r} (-1)^{i+1} \left(a\cdot a_i\right)\ a_1 \wedge \cdots \wedge a_{i-1} \wedge a_{i+1} \wedge \cdots \wedge a_r \\ a\wedge A_r &= \frac{1}{2}\left[aA_r + (-1)^r A_r a\right] = a\wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r \end{aligned}$$

Relationships Among Products

Pseudoscalar

$$\boldsymbol{i} = \boldsymbol{e}_0 \wedge \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_{D-1} \qquad \boldsymbol{i}^2 = (-1)^{\frac{D(D-1)}{2}} g_{00} \cdots g_{D-1,D-1}$$

$$\boldsymbol{i}\left[\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_n}\right] = g_{i_1 i_1} \cdots g_{i_n i_n} \frac{1}{n!} \sum_{i_1 \cdots i_n} \epsilon_{i_1 \cdots i_n}^{i_{n+1} \cdots i_D} \left[\boldsymbol{e}_{i_{n+1}} \wedge \cdots \wedge \boldsymbol{e}_{i_D}\right]$$

Symmetry

$$\boldsymbol{e}_k \boldsymbol{i} = \boldsymbol{e}_k \boldsymbol{e}_0 \boldsymbol{e}_1 \cdots \boldsymbol{e}_k \cdots \boldsymbol{e}_{D-1} = (-1)^{k-1} \boldsymbol{e}_0 \boldsymbol{e}_1 \cdots \boldsymbol{e}_k \boldsymbol{e}_k \cdots \boldsymbol{e}_{D-1} = (-1)^{D-1} \boldsymbol{i} \boldsymbol{e}_k$$

Useful Identity

$$\boldsymbol{i}\left(a \wedge B_r\right) = \frac{1}{2}\left[\boldsymbol{i} a B_r + (-1)^r \boldsymbol{i} B_r a\right]$$

$$\begin{aligned} a \cdot \left(\boldsymbol{i} B_r\right) &= \frac{1}{2}\left[a \boldsymbol{i} B_r - (-1)^{D-r} \boldsymbol{i} B_r a\right] \\ &= \frac{1}{2}\left[(-1)^{D-1} \boldsymbol{i} a B_r - (-1)^{D-r} \boldsymbol{i} B_r a\right] \\ &= (-1)^{D-1} \frac{1}{2}\left[\boldsymbol{i} a B_r + (-1)^r \boldsymbol{i} B_r a\right] \\ &= (-1)^{D-1} \boldsymbol{i}\left(a \wedge B_r\right) \end{aligned}$$

Formulation of Field Equations

Definitions

$$\boldsymbol{d} = \partial^{\alpha}\mathbf{e}_{\alpha} \qquad \boldsymbol{f} = 1/2 f^{\alpha\beta}\mathbf{e}_{\alpha} \wedge \mathbf{e}_{\beta}$$

$$\boldsymbol{j_e} = j^{\alpha}\mathbf{e}_{\alpha} \qquad \mathbf{e}_{\alpha} \cdot \mathbf{e}_{\beta} = g_{\alpha\beta}$$

Field Equations

$$\boldsymbol{df} = -e\boldsymbol{j_e} \quad \Rightarrow \quad \boldsymbol{d}\cdot\boldsymbol{f} + \boldsymbol{d}\wedge\boldsymbol{f} = -e\boldsymbol{j}_e$$

$$\boldsymbol{d}\cdot\boldsymbol{f} = -e\boldsymbol{j}_e \quad \rightarrow \quad \frac{1}{2}\left(\partial^{\gamma}\mathbf{e}_{\gamma}\right)\cdot\left(f^{\alpha\beta}\mathbf{e}_{\alpha}\wedge\mathbf{e}_{\beta}\right) = \frac{1}{2}\partial^{\gamma} f^{\alpha\beta}\mathbf{e}_{\gamma}\cdot\left(\mathbf{e}_{\alpha}\wedge\mathbf{e}_{\beta}\right)$$

$$= \frac{1}{2}\partial^{\gamma} f^{\alpha\beta}\left(g_{\gamma\alpha}\mathbf{e}_{\beta} - g_{\gamma\beta}\mathbf{e}_{\alpha}\right)$$

$$= \partial_{\alpha} f^{\alpha\beta}\mathbf{e}_{\beta} = j^{\beta}\mathbf{e}_{\beta}$$

$$\boldsymbol{d}\wedge\boldsymbol{f} = 0 \quad \rightarrow \quad \boldsymbol{id}\wedge\boldsymbol{f} = \boldsymbol{d}\cdot\boldsymbol{if} = \left(\partial^{\alpha}\mathbf{e}_{\alpha}\right)\cdot\left[1/(D-2)!\,\epsilon^{\alpha_1\alpha_2\ldots\alpha_{D-2}\beta\gamma} f_{\beta\gamma}\mathbf{e}_{\alpha_1}\mathbf{e}_{\alpha_2}\ldots\mathbf{e}_{\alpha_{D-2}}\right]$$

$$= \left[1/(D-2)!\,\epsilon^{\alpha_1\alpha_2\ldots\alpha_{D-3}\alpha\beta\gamma}\partial_{\alpha} f_{\beta\gamma}\mathbf{e}_{\alpha_1}\mathbf{e}_{\alpha_2}\ldots\mathbf{e}_{\alpha_{D-3}}\right] = 0$$

D = 4

$$\partial_{\nu}F^{\mu\nu} = eJ^{\mu} \qquad \partial_{\mu}\epsilon^{\mu\nu\rho\lambda}F_{\nu\rho} = 0$$

D = 5

$$\partial_{\beta} f^{\alpha\beta} = ej^{\alpha} \qquad \partial_{\alpha}\epsilon^{\alpha\beta\gamma\delta\epsilon} f_{\beta\gamma} = 0$$

Duality and the Dirac Monopole

Suppose a magnetic current

$$\nabla \cdot \mathbf{E} = eJ_e^0 \qquad \nabla \times \mathbf{H} - \partial_0 \mathbf{E} = e\mathbf{J}_e$$

$$\nabla \cdot \mathbf{H} = gJ_m^0 \qquad \nabla \times \mathbf{E} + \partial_0 \mathbf{H} = g\mathbf{J}_m$$

Perform duality rotation

$$\mathbf{E} = \mathbf{E}'\cos\theta + \mathbf{H}'\sin\theta \qquad \mathbf{H} = -\mathbf{E}'\sin\theta + \mathbf{H}'\cos\theta$$

$$e\mathbf{J}_e = e\mathbf{J}_e'\cos\theta + g\mathbf{J}_m'\sin\theta \qquad eJ_e^0 = eJ_e^0{}'\cos\theta + gJ_m^0{}'\sin\theta$$

$$g\mathbf{J}_m = -e\mathbf{J}_e'\sin\theta + g\mathbf{J}_m'\cos\theta \qquad eJ_e^0 = -eJ_e^0{}'\sin\theta + gJ_m^0{}'\cos\theta$$

Choose angle so that

$$\nabla \cdot \mathbf{E}' = eJ_e^0 \qquad \nabla \times \mathbf{H}' - \partial_0 \mathbf{E}' = e\mathbf{J}_e'$$

$$\nabla \cdot \mathbf{H}' = 0 \qquad \nabla \times \mathbf{E}' + \partial_0 \mathbf{H}' = 0$$

Quantization condition

$$\frac{ge}{\hbar c} = \frac{n}{2}$$

Magnetic monopole $\Rightarrow$ Charge quantization

Equivalent duality rotation

$$\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}(\mathbf{E},\mathbf{H}) = F^{\mu\nu}(-\mathbf{H},\mathbf{E})$$

$$F^{\mu\nu} \to F^{\mu\nu}\cos\theta + \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}\sin\theta \qquad \epsilon^{\mu\nu\lambda\rho} F^{\mu\nu} \to -F^{\mu\nu}\sin\theta + \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}\cos\theta$$

$$eJ_e^\mu \to eJ_e^\mu\cos\theta + gJ_m^\mu\sin\theta \qquad gJ_m^\mu = -eJ_e^\mu{}'\sin\theta + gJ_m^\mu{}'\cos\theta$$

Duality Transformation

General field equations	$\boldsymbol{df} = \boldsymbol{j} = -e\boldsymbol{j}_1 - g\boldsymbol{j}_3$
Electric current equation	$\boldsymbol{d} \cdot \boldsymbol{f} = -e\boldsymbol{j}_1$
Magnetic current equation	$\boldsymbol{d} \wedge \boldsymbol{f} = -g\boldsymbol{j}_3$ $\boldsymbol{d} \wedge \boldsymbol{ih} = \boldsymbol{id} \cdot \boldsymbol{h} = -g\boldsymbol{iq}_{D-3} \quad \rightarrow \quad \boldsymbol{d} \cdot \boldsymbol{h} = -g\boldsymbol{q}_{D-3}$

1-parameter transformation

$$U = e^{\theta G} = 1 + \theta G + o(\theta^2)$$

$$U\boldsymbol{df} = U\boldsymbol{j}$$

$$(1 + \theta G)(\boldsymbol{d} \cdot \boldsymbol{f} + \boldsymbol{d} \wedge \boldsymbol{f}) = (1 + \theta G)(-e\boldsymbol{j}_1 - g\boldsymbol{j}_3)$$

$$\boldsymbol{d} \cdot \boldsymbol{f} + \boldsymbol{d} \wedge \boldsymbol{f} + \theta G\,(\boldsymbol{d} \cdot \boldsymbol{f}) + \theta G\,(\boldsymbol{d} \wedge \boldsymbol{f}) = \left(-e\boldsymbol{j}_1 - g\boldsymbol{j}_3 - e\theta G\boldsymbol{j}_1 - g\theta G\boldsymbol{j}_3\right)$$

Require separation by rank

Vector equation $\quad \boldsymbol{d} \cdot \boldsymbol{f} + \theta G\,(\boldsymbol{d} \wedge \boldsymbol{f}) = \left(-e\boldsymbol{j}_1 - g\theta G\boldsymbol{j}_3\right)$

Trivector equation $\quad \boldsymbol{d} \wedge \boldsymbol{f} + \theta G\,(\boldsymbol{d} \cdot \boldsymbol{f}) = \left(-g\boldsymbol{j}_3 - e\theta G\boldsymbol{j}_1\right)$

Duality Symmetry

Require form invariance

$$\left.\begin{aligned} G(\boldsymbol{d}\wedge\boldsymbol{f}) &= \boldsymbol{d}\cdot(G\boldsymbol{f}) \\ G(\boldsymbol{d}\cdot\boldsymbol{f}) &= \boldsymbol{d}\wedge(G\boldsymbol{f}) \end{aligned}\right\} \Rightarrow \begin{cases} \boldsymbol{d}\cdot(\boldsymbol{f}+\theta G\boldsymbol{f}) = -e\boldsymbol{j}_1 - g\theta G\boldsymbol{j}_3 \\ \boldsymbol{d}\wedge(\boldsymbol{f}+\theta G\boldsymbol{f}) = -g\boldsymbol{j}_3 - e\theta G\boldsymbol{j}_1 \end{cases}$$

Requirements on G

$\boldsymbol{j}_1$ and $G\boldsymbol{j}_3$ must be vectors $\quad (G: 3 \to 1)$

$\boldsymbol{j}_3$ and $G\boldsymbol{j}_1$ must be trivectors $\quad (G: 1 \to 3)$

$G\boldsymbol{f}$ must be a bivector $\quad (G: 2 \to 2)$

$$G(\boldsymbol{d}\cdot\boldsymbol{f}) = \boldsymbol{d}\wedge(G\boldsymbol{f}) \Rightarrow G\boldsymbol{d} = -\boldsymbol{d}G$$

Requirements are only satisfied for $G = -\boldsymbol{i}$ in $D = 4$

Choose θ so that

$$\left.\begin{aligned} \boldsymbol{d}\cdot(\boldsymbol{f}-\theta\boldsymbol{i}\boldsymbol{f}) &= \boldsymbol{d}\cdot\boldsymbol{f}' \\ -e\boldsymbol{j}_1 - g\theta\boldsymbol{i}\boldsymbol{j}_3 &= -e\boldsymbol{j}_1' \end{aligned}\right\} \to \boldsymbol{d}\cdot\boldsymbol{f}' = -e\boldsymbol{j}_1'$$

$$\left.\begin{aligned} \boldsymbol{d}\wedge(\boldsymbol{f}-\theta\boldsymbol{i}\boldsymbol{f}) &= \boldsymbol{d}\wedge\boldsymbol{f}' \\ -g\boldsymbol{j}_3 - e\theta\boldsymbol{i}\boldsymbol{j}_1 &= 0 \end{aligned}\right\} \to \boldsymbol{d}\wedge\boldsymbol{f}' = 0$$

$$U = e^{-i\theta}$$

$$U\boldsymbol{d}\boldsymbol{f} = \boldsymbol{d}U\boldsymbol{f} = \boldsymbol{d}\boldsymbol{f}'$$

$$U\boldsymbol{j} = U(-e\boldsymbol{j}_1 - g\boldsymbol{j}_3) = -e\boldsymbol{j}_1'$$

Generalized Electromagnetic Fields in D=5

Multivector fields and currents

$$dF = J$$

$$F = F_0 + F_1 + F_2 + F_3 + F_4 + F_5$$

$$J = J_0 + J_1 + J_2 + J_3 + J_4 + J_5$$

Separating Terms

$$d \cdot F_1 = J_0 \qquad d \wedge F_2 + d \cdot F_4 = J_3$$

$$dF_0 + d \cdot F_2 = J_1 \qquad d \wedge F_3 + d \cdot F_5 = J_4$$

$$d \wedge F_1 + d \cdot F_3 = J_2 \qquad d \wedge F_4 = J_5$$

Duality Transformation

$$U = 1 + \theta \boldsymbol{i} + o(\theta^2)$$

$$d \cdot F_1 + \theta d \cdot iF_4 = J_0 + \theta i J_5$$

$$dF_0 + d \cdot F_2 + \theta d \cdot iF_3 + \theta i d \cdot F_5 = J_1 + \theta i J_4$$

$$d \wedge F_1 + d \cdot F_3 + \theta d \cdot iF_2 + \theta d \wedge iF_4 = J_2 + \theta i J_3$$

$$d \wedge F_2 + d \cdot F_4 + \theta d \cdot iF_1 + \theta d \wedge iF_3 = J_3 + \theta i J_2$$

$$d \wedge F_3 + d \cdot F_5 + \theta d \wedge iF_2 = J_4 + \theta i J_1$$

$$d \wedge F_4 + \theta i dF_0 + \theta d \wedge iF_1 = J_5 + \theta i J_0$$

Identify Associated Terms

$$d \cdot F_1 + \theta d \cdot iF_4 = J_0 + \theta iJ_5 \rightarrow d \cdot \left(F_1 + \theta iF_4\right) = \boxed{d \cdot \tilde{F}_1 = \tilde{J}_0}$$

$$d \cdot F_2 + \theta d \cdot iF_3 + \theta id \cdot F_5 = J_1 + \theta iJ_4 \rightarrow d \cdot \left(F_2 + \theta iF_3\right) = \boxed{d \cdot \tilde{F}_2 = \tilde{J}_1}$$

$$d \wedge F_1 + d \cdot F_3 + \theta d \cdot iF_2 + \theta d \wedge iF_4 = J_2 + \theta iJ_3$$

$$d \wedge \left(F_1 + \theta iF_4\right) + d \cdot \left(F_3 + \theta iF_2\right) = \boxed{d \wedge \tilde{F}_1 + d \cdot \tilde{F}_3 = \tilde{J}_2}$$

$$d \wedge F_2 + d \cdot F_4 + \theta d \cdot iF_1 + \theta d \wedge iF_3 = J_3 + \theta iJ_2$$

$$d \wedge \left(F_2 + \theta iF_3\right) + d \cdot \left(F_4 + \theta iF_1\right) = \boxed{d \wedge \tilde{F}_2 + d \cdot \tilde{F}_4 = \tilde{J}_3}$$

$$d \wedge F_3 + \theta d \wedge iF_2 = J_4 + \theta iJ_1 \rightarrow d \wedge \left(F_3 + \theta iF_2\right) = \boxed{d \wedge \tilde{F}_3 = \tilde{J}_4}$$

$$d \wedge F_4 + \theta d \wedge iF_1 = J_5 + \theta iJ_0 \rightarrow d \wedge \left(F_4 + \theta iF_1\right) = \boxed{d \wedge \tilde{F}_4 = \tilde{J}_5}$$

Minimum Set of Fields

Generalized field equations

$$d \cdot \tilde{F}_1 = \tilde{J}_0 \qquad d \wedge \tilde{F}_2 + d \cdot \tilde{F}_4 = \tilde{J}_3$$

$$d \cdot \tilde{F}_2 = \tilde{J}_1 \qquad d \wedge \tilde{F}_3 = \tilde{J}_4$$

$$d \wedge \tilde{F}_1 + d \cdot \tilde{F}_3 = \tilde{J}_2 \qquad d \wedge \tilde{F}_4 = \tilde{J}_5$$

Eliminate scalar and pseudoscalar current sectors

Set magnetic current to zero

$$d \cdot \tilde{F}_2 = \tilde{J}_1 \qquad d \wedge \tilde{F}_2 = \tilde{J}_3 = 0$$

$$d \cdot \tilde{F}_3 = \tilde{J}_2 \qquad d \wedge \tilde{F}_3 = \tilde{J}_4$$

Standard field equations <u>without</u> magnetic current

$$\boxed{d \cdot \widetilde{F}_2 = \tilde{J}_1 \qquad d \wedge \widetilde{F}_2 = 0}$$

Equivalent field equations <u>with</u> magnetic current

$$d \cdot \tilde{F}_3 = d \cdot iH_2 = \tilde{J}_2 \quad \rightarrow \quad \boxed{\begin{array}{c} d \wedge H_2 = i\tilde{J}_2 = Q_3 \\ d \cdot H_2 = i\tilde{J}_4 = Q_1 \end{array}}$$

Full Set of Fields

Generalized field equations	$d\cdot\tilde{F}_1=\tilde{J}_0$ $d\wedge\tilde{F}_1+d\cdot\tilde{F}_3=\tilde{J}_2$ $d\wedge\tilde{F}_3=\tilde{J}_4$	$d\cdot\tilde{F}_2=\tilde{J}_1$ $d\wedge\tilde{F}_2+d\cdot\tilde{F}_4=\tilde{J}_3$ $d\wedge\tilde{F}_4=\tilde{J}_5$
2-field sector <u>without</u> magnetic current	$d\cdot\tilde{F}_2=\tilde{J}_1$	$d\wedge\tilde{F}_2=0$
4-field	$d\cdot\tilde{F}_4=\tilde{J}_3\rightarrow d\wedge H_1=Q_2$	$d\wedge\tilde{F}_4=\tilde{J}_5\rightarrow d\cdot H_1=Q_0$
3-field and 1-field	$d\wedge\tilde{F}_1+d\cdot\tilde{F}_3=\tilde{J}_2\rightarrow d\cdot H_4+d\wedge H_2=Q_3$ $d\wedge\tilde{F}_3=\tilde{J}_4\rightarrow d\cdot H_2=Q_1$	$d\cdot\tilde{F}_1=\tilde{J}_0\rightarrow d\wedge H_4=Q_5$
Equivalent 2-field sector <u>with</u> magnetic current	$d\cdot H_2=Q_1$ $d\cdot H_4+d\wedge H_2=Q_3$	$d\wedge H_4=Q_5$
Equivalent 1-field sector	$d\cdot H_1=Q_0$	$d\wedge H_1=Q_2$

Lorentz Force With Magnetic Charges

$$\boldsymbol{F} = e\left[\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{H}\right] + g\left[\boldsymbol{H} - \boldsymbol{v}\times\boldsymbol{E}\right]$$

Modified Lorentz force in D=4

$$M\frac{du^i}{dt} = e\left[E^i + \epsilon^{ijk}v_j \times H_k\right] + g\left[H^i - \epsilon^{ijk}v_j \times E_k\right]$$

$$M\frac{du^i}{dt}\frac{dt}{d\tau} = e\left[E^i\frac{dt}{d\tau} + \epsilon^{ijk}u_j \times H_k\right] + g\left[H^i\frac{dt}{d\tau} - \epsilon^{ijk}u_j \times E_k\right]$$

$$= e\left[F^{i0}\dot{x}_0 + F^{ij}u_j\right] + g\epsilon^{0i\mu\nu}\left[F_{\mu\nu}\dot{x}_0 + F_{0\mu}\dot{x}_\nu\right]$$

$$M\frac{d^2x^i}{d\tau^2} = eF^{i\mu}\dot{x}_\mu + g\epsilon^{i\mu\nu\lambda}F_{\mu\nu}\dot{x}_\lambda$$

$$= eF^{i\mu}\dot{x}_\mu + g\widetilde{F}^{i\mu}\dot{x}_\mu \xrightarrow[\text{duality rotation}]{} e\left(F'\right)^{i\mu}\dot{x}_\mu$$

---

no duality rotation in D=5

$$M\frac{d^2x^\mu}{d\tau^2} = ef^{\mu\alpha}\dot{x}_\alpha + g\tilde{f}^{\mu\alpha}\dot{x}_\alpha$$

$$= ef^{\mu\alpha}\dot{x}_\alpha + g\epsilon^{\mu\alpha\beta\gamma\delta}f_{\alpha\beta\gamma}\dot{x}_\delta$$